SOME PROBLEMS ON OE and vNE of GROUP ACTIONS

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The von Neumann algebra $L^\infty(X) \rtimes \Gamma$
(the **Group Measure Space Construction**)
of a group action $\Gamma \curvearrowright (X, \mu)$

- $\mathcal{H} = \oplus_g L^2(X) u_g$ Hilbert space
- $\mathcal{H} \ni \Sigma_h \xi_h u_h$ "Fourier-like" series
- $L^2(X) \ni \xi_h$ "coefficients"
- $u_h, h \in \Gamma$, copy of Γ
- Multiplication: $(x_g u_g)(\xi_h u_h) = x_g g(\xi_h) u_{gh}$

$$L^\infty(X) \rtimes \Gamma \stackrel{\text{def}}{=} \{x \in \mathcal{H} \mid x\xi \in \mathcal{H}, \forall \xi \in \mathcal{H}\}$$

Case $\Gamma \curvearrowright \{\cdot\}$ gives the
Group vN algebra $\mathcal{L}(\Gamma) \stackrel{\text{def}}{=} \mathbb{C} \rtimes \Gamma$

As algebras of left mult. operators on $\mathcal{H}$, they are *von Neumann algebras*, i.e. closed in topology given by seminorms $|\langle \cdot \xi, \eta \rangle|$ on $\mathcal{B}(\mathcal{H})$.

• $L^\infty(X)$ as subalgebra of $M = L^\infty(X) \rtimes \Gamma$ by $a \mapsto au_e = a1$

• $\int \cdot \mathrm{d}\mu$ extends to positive linear functional τ on M by $\tau(\Sigma_g x_g u_g) = \int x_e \mathrm{d}\mu$.
Satisfies $\tau(xy) = \tau(yx)$, $\forall x, y$,
i.e. τ *trace* on M. [Obs $\tau(y^* x) = \langle x, y \rangle_{\mathcal{H}}$]

• $\Gamma \curvearrowright (X, \mu)$ free ergodic, $|\Gamma| = \infty$, then M II_1 *factor*, i.e. $\mathcal{Z}(M) = \mathbb{C}$, M has unique trace, $\tau(\mathcal{P}(M)) = [0,1]$ (*"continuous dimension"*), with $L^\infty(X) \subset M$ maximal abelian, called *Cartan subalgebra*

• $\mathcal{L}(\Gamma)$ is II_1 factor iff Γ *infinite conjugacy class* (*ICC*). E.g.: $\Gamma = S_\infty, \mathbb{F}_n, PSL(n, \mathbb{Z})$, $n \geq 2$.

- *Continuous dimension* allows *t-amplification* of II_1 factor M, $\forall t > 0$, by $M^t = pM_{n\times n}(M)p$, $n \geq t$, $p \in \mathcal{P}(M_{n\times n}(M))$, $\tau(p) = t/n$.
Notice: $(M^t)^s = M^{ts}$.

- *Fundamental group* of M:
$\mathcal{F}(M) \overset{\text{def}}{=} \{t > 0 \mid M^t \simeq M\}$.

Conjugacy of $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ means $\Delta : (X, \mu) \simeq (Y, \nu)$ and $\delta : \Gamma \simeq \Lambda$
with $\Delta(gt) = \delta(g)\Delta(t)$, $\forall g \in \Gamma, t \in X$.

Note: Conjugacy implements isomorphism
$L^\infty(X) \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ by $\Sigma a_g u_g \mapsto \Sigma \Delta(a_g) v_{\delta(g)}$

Singer '55: *$L^\infty(X) \rtimes \Gamma$ can only "remember" the equivalence relation given by orbits of $\Gamma \curvearrowright X$: $\mathcal{R}_\Gamma \stackrel{\text{def}}{=} \{(t, gt) \mid t \in X, g \in \Gamma\}$*

Feldman-Moore '77: *An iso $\Delta : (X, \mu) \simeq (Y, \nu)$ extends to $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ iff Δ is an* orbit equivalence *(*OE*), i.e. $\Delta(\mathcal{R}_\Gamma) = \mathcal{R}_\Lambda$, or $\Delta(\Gamma t) = \Lambda\Delta(t)$, $\forall t$.*

Construction of von Neumann algebra $\mathcal{L}(\mathcal{R}_\Gamma)$ of *equivalence relation* $\mathcal{R}_\Gamma$ associated with an action $\Gamma \curvearrowright X$. It is II_1 factor if $\Gamma \curvearrowright X$ ergodic. $L^\infty(X) \subset \mathcal{L}(\mathcal{R}_\Gamma)$ is maximal abelian. $\mathcal{L}(\mathcal{R}_\Gamma)$, $L^\infty(X) \rtimes \Gamma$ coincide when action is free.

Obs: *Conjugacy $\Rightarrow$ OE $\Rightarrow$ iso of vN algebras* (or *vNE*)

A *deformation* of II_1 factor $M = L^\infty(X) \rtimes \Gamma$, $\mathcal{L}(\Gamma)$ is a sequence of *completely positive definite* (c.p.) maps $\phi_n : M \to M$ which are unital, trace preserving and satisfy $\lim_n \|\phi_n(x) - x\|_2 = 0$, $\forall x \in M$.

Examples of c.p. maps:

- Automorphisms of M;

- Maps of the form $\phi(\Sigma_g a_g u_g) = \Sigma_g \varphi(g) a_g u_g$, where $\varphi : \Gamma \to \mathbb{C}$ is positive definite.

- **Problems on relative property (T)**

A subalgebra $B \subset M$ has relative property (T) if any deformation ϕ_n of M is uniform on B: $\lim_n(\sup\{\|\phi_n(b) - b\|_2 \mid b \in (B)_1\}) = 0$.

An action $\Gamma \curvearrowright (X, \mu)$ (resp. its eq. rel. $\mathcal{R}_\Gamma$) has relative property (T) if $L^\infty(X) \subset \mathcal{L}(\mathcal{R}_\Gamma)$ has relative property (T).

Obs: *If H discrete abelian group and $\Gamma \curvearrowright H$ then $H \subset H \rtimes \Gamma$ has rel. prop.* (T) *iff $\Gamma \curvearrowright \hat{H}$ has rel. prop.* (T);

Examples $SL(n, \mathbb{Z}) \curvearrowright \mathbb{Z}^n$, $n \geq 2$ (Kazhdan); $\Gamma \curvearrowright \mathbb{Z}^2$ for $\Gamma \subset SL(2, \mathbb{Z})$ non-amenable (Burger); Shalom, Fernos, Valette.

Problem 1 Give a "non-vN Alagebra" def. of relative property (T) for actions.

Problem 2 What are the groups Γ for which $\exists\, \Gamma \curvearrowright (X, \mu)$ free ergodic with rel. prop. (T)

- **Related Questions:**

Fact ([P01]) If $\Gamma \curvearrowright X$ ergodic has rel prop (T) then Out($\mathcal{R}_\Gamma$), $\mathcal{F}(\mathcal{R}_\Gamma)$ are countable and $\mathcal{R}_\Gamma$ has only countably many quotients (...).

Problem 3 Calculate Out($\mathcal{R}_{\Gamma \curvearrowright \mathbb{Z}^2}$) for $\Gamma = SL(2, \mathbb{Z})$ and $\Gamma = \mathbb{F}_n \subset SL(2, \mathbb{Z})$.

Problem 4 Show $\exists \mathbb{F}_n \curvearrowright X$ free ergodic with Out($\mathcal{R}_{\mathbb{F}_n}$) $= 1$.

Problem 5 $\exists \mathbb{F}_\infty \curvearrowright X$ free ergodic with $\mathcal{F}(\mathcal{R}_{\mathbb{F}_\infty}) = 1$, resp. $\mathcal{F}(\mathcal{R}_{\mathbb{F}_\infty}) = \mathbb{R}^*_+$? (By Gaboriau, $\mathcal{F}(\mathcal{R}_{\mathbb{F}_n}) = 1$ for any free ergodic action of $\mathbb{F}_n$, $n < \infty$).

Problem 6 Construct $\Gamma \curvearrowright X$ with $\mathcal{R}_\Gamma$ having no (finite index) quotients $\theta : X \to Y$ with θ iso (1 to 1) on each orbit of $\mathcal{R}_\Gamma$ (Vaes).

Problem 7 Show that any $\mathcal{F}(\mathcal{R}_\Gamma)$ (resp. $\mathcal{F}(\mathcal{L}(\mathcal{R}_\Gamma))$) is either countable or $\mathbb{R}^*_+$.

- **Some OE "superrigidity" problems**

Fact ([P05, P06]) If Γ ICC Kazhdan then Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ are $\mathcal{U}_{fin}$-Cocycle Superrigid, in particular OE Superrigid, i.e. any OE with another free action $\Lambda \curvearrowright Y$ comes from a conjugacy. Also true for *sub-malleable mixing actions* (e.g. Gaussians and their quotients) $\Gamma \curvearrowright X$ with:

- Γ having infinite w-normal subgroup with rel prop (T)

- Γ having a non-amenable subgroup with infinite centralizer.

Problem 1 Find the class $\mathcal{CS}$ (resp. $\mathcal{OES}$) of groups Γ such that any Bernoulli Γ-action is $\mathcal{U}_{fin}$-Cocycle (resp OE) Superrigid.

Problem 2 Find larger classes $\mathcal{U}$ of "target" groups with the property that any Bernoulli action of a Kazhdan (or other) group is $\mathcal{U}$-Cocycle Superrigid.

Problem 3 Calculate $\mathsf{H}^2(\mathcal{R}_\Gamma)$ for some actions $\Gamma \curvearrowright X$, more generally $\mathsf{H}^n(\mathcal{R}_\Gamma)$ (no such calculations exist for $n \geq 2$!).

Problem 4 Is it true that $\forall \Gamma, \Lambda$ non-amenable, any OE of Bernoulli actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ comes from a conjugacy ? For free groups ?

Problem 5 Let Γ be an ICC Kazhdan group (or other "special" non-amenable). Is any automorphism of the probability space $(X, \mu) = (X_0, \mu_0)^\Gamma$ that commutes with the Bernoulli action $\Gamma \curvearrowright X$ the product of a diagonal automorphism and a "right" Bernoulli shift by an element of the group?

- **Some vNE "superrigidity" problems**

Fact ([P05, P06]) If Γ, Λ ICC groups, with Γ either Kazhdan, or having w-normal subgroup with rel prop (T), or having a non-amenable subgroup with infinite centralizer, and $\Gamma \curvearrowright X$ is free mixing while $\Lambda \curvearrowright Y$ is Bernoulli, then any isomorphism $\theta : L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ comes from a conjugacy (Strong vNE Rigidity).

Problem 1 Find classes of group actions $\Gamma \curvearrowright X$ that are vNE Superrigid, i.e. given any other free ergodic action $\Lambda \curvearrowright Y$, any isomorphism $\theta : L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ comes from a conjugacy.

Problem 2 Find classes of factors $L^\infty(X) \rtimes \Gamma$ with unique Cartan subalgebras (...).

Obs If in the results in ([P05,06]) one could prove that any two Cartan subalgebras of $L^\infty(X) \rtimes \Gamma$ are conjugate, for Γ Kazhdan (or other class of groups) and $\Gamma \curvearrowright X$ Bernoulli, then $\Gamma \curvearrowright X$ follows vNE Superrigid.

Problem 3 Show that if $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ with Γ Kazhdan and $\Gamma \curvearrowright X$ Bernoulli, then Λ follows Kazhdan. (OBS: If so, then Bernoulli actions of Kazhdan groups follow vNE Superrigid).

Problem 4 Show that the factor M associated with $PSL(n, \mathbb{Z}) \curvearrowright \mathbb{Z}^n$ has unique Cartan. (Obs: By Furman, the action would then follow vNE Superrigid).

- **Connes' Rigidity Conjecture '80:** If Γ, Λ ICC groups with property (T), $\mathcal{L}(\Gamma) \simeq \mathcal{L}(\Lambda) \Rightarrow \Gamma \simeq \Lambda$? At least for $PSL(n, \mathbb{Z})$, $n \geq 3$? Even: If $\theta : L(\Gamma) \simeq \mathcal{L}(\Lambda)$ then $\exists \delta : \Gamma \to \Lambda$ and $\gamma \in Hom(\Gamma, \mathbb{T})$ such that $\theta(\Sigma_g c_g u_g) = \Sigma_g \gamma(g) c_g u_{\delta(g)}$?

- **The Free Group Factor Problem**:
If $2 \leq n, m \leq \infty$, does $\mathcal{L}(\mathbb{F}_n) \simeq \mathcal{L}(\mathbb{F}_m)$ imply $n = m$? Sufficient to prove: $\mathcal{L}(\mathbb{F}_\infty) \neq \mathcal{L}(\mathbb{F}_n)$ for some n (cf. Voiculescu, Radulescu, Dykema). Related to this:

- **Finite Generation Problem** Can $\mathcal{L}(\mathbb{F}_\infty)$ be finitely generated as vN Alg ? Do there exist $\mathcal{L}(\Gamma)$ which cannot be finitely generated as vN alg ? (Obs: Any factor $\mathcal{L}(\mathcal{R}_\Gamma)$ can be generated by two unitaries)