Popa's rigidity theorems and II_1 factors without non-trivial finite index subfactors

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Talk in two parts

1. Sorin Popa's cocycle superrigidity theorems.

   $\rightsquigarrow$ Sketch of proof.

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   ⤳ Sketch of proof.

   ⤳ Needing von Neumann algebras.

2. Application : bimodules of certain II_1 factors.

   ⤳ Kind of representation theory.

Group actions and 1-cocycles

Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving action.

Standing assumptions : essentially free and ergodic.

Definition

A 1-cocycle for $\Gamma \curvearrowright X$ with values in a Polish group $\mathcal{V}$, is a measurable map

$$\omega : X \times \Gamma \to \mathcal{V}$$

satisfying $\omega(x, gh) = \omega(x, g)\omega(x \cdot g, h)$

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- $\omega_1 \sim \omega_2$ if $\exists \varphi$ with $\omega_1(x, g) = \varphi(x)\omega_2(x, g)\varphi(x \cdot g)^{-1}$
- Homomorphisms $\Gamma \to \mathcal{V}$: 1-cocycles not depending on $x \in X$.
- (Zimmer) Orbit equivalence $\rightsquigarrow$ 1-cocycle.

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Let $\mathcal{U}$ be a class of Polish groups.

Definition

$\Gamma \curvearrowright (X, \mu)$ is $\mathcal{U}$-cocycle superrigid if every 1-cocycle for $\Gamma \curvearrowright X$ with values in a group $\mathcal{V} \in \mathcal{U}$, is cohomologous to a homomorphism.

Statement of Popa's cocycle superrigidity

Generalized Bernoulli action : let $\Gamma \curvearrowright I$ with I a countable set and set

$$\Gamma \curvearrowright (X, \mu) := \prod_I (X_0, \mu_0) \ .$$

$\mathcal{U}$: class containing all compact and all discrete groups.

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Theorem (Popa, 2005-2006)

Let $\Gamma \curvearrowright (X,\mu)$ be a generalized Bernoulli action and $H \lhd \Gamma$ a normal subgroup with $H \cdot i$ infinite for all $i \in I$.

In both of the following cases, $\Gamma \curvearrowright X$ is $\mathcal{U}$-cocycle superrigid.

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2. There exists a non-amenable $H' < \Gamma$, centralizing H and with $H' \curvearrowright I$ having amenable stabilizers.

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1. $H \subset \Gamma$ has the relative property (T).
2. There exists a non-amenable $H' < \Gamma$, centralizing H and with $H' \curvearrowright L^2(X)$ having stable spectral gap.

Definition

A von Neumann algebra is
a weakly closed unital $*$-subalgebra of $B(H)$.

Background : von Neumann algebras

Examples of von Neumann algebras

- $B(H)$ itself.
- $L^\infty(X,\mu)$ (as acting on $L^2(X,\mu)$).
- The group von Neumann algebra $\mathcal{L}(\Gamma)$ generated by unitaries λ_g on $\ell^2(\Gamma)$: $\lambda_g \delta_h = \delta_{gh}$.

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Finite von Neumann algebras : admitting tracial state τ.

Finite von Neumann alg. (M,τ) $\overset{\text{GNS}}{\rightsquigarrow}$ Hilbert space $L^2(M,\tau)$ which is an M-M-bimodule.

Example

- $\ell^2(\Gamma)$ is an $\mathcal{L}(\Gamma)$-$\mathcal{L}(\Gamma)$-bimodule : $\lambda_g \delta_h \lambda_k = \delta_{ghk}$.
- $\mathcal{L}(\Gamma) \hookrightarrow \ell^2(\Gamma)$ densely : $x \mapsto x\delta_e$.

Special case of Popa's theorem : sketch of proof

Theorem

Let Γ be a property (T) group and $\Gamma \curvearrowright I$ with infinite orbits.

Take $\Gamma \curvearrowright (X, \mu) := \prod_I [0,1]$.

Every 1-cocycle $\omega : X \times \Gamma \to \Lambda$ with values in the countable group Λ is cohomologous to a homomorphism.

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First ingredient : property (T).

Second ingredient : Popa's malleability.

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$\leadsto$ There exist a flow $(\alpha_t)_{t\in\mathbb{R}}$ and an involutive β on $X \times X$:

- α_t and β commute with the diagonal Γ-action,
- $\alpha_1(x, y) = (y, \ldots)$
- $\beta\alpha_t\beta = \alpha_{-t}$ and $\beta(x, y) = (x, \ldots)$

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Also : $\Gamma \curvearrowright (X, \mu)$ is weakly mixing.

Sketch of the proof

Take $\omega : X \times \Gamma \to \Lambda$ and define

$$\omega_0 : X \times X \times \Gamma \to \Lambda : \omega_0(x, y, g) = \omega(x, g)$$
$$\omega_t : X \times X \times \Gamma \to \Lambda : \omega_t(x, y, g) = \omega_0(\alpha_t(x, y), g) \ .$$

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$\rightsquigarrow$ New actions : $\Gamma \curvearrowright X \times X \times \Lambda$:
$(x, y, s) \cdot g = (x \cdot g \ , \ y \cdot g \ , \ \omega_t(x, y, g)^{-1} \, s \, \omega_0(x, y, g))$

$\rightsquigarrow$ Unitary representations : $\pi_t : \Gamma \to \mathcal{U}(L^2(X \times X \times \Lambda))$.

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Property (T) yields $t = 1/n$ and $\varphi \in L^2(X \times X, \ell^2(\Lambda))$ with

$$\omega_{1/n}(x, y, g) \; \varphi(x \cdot g, y \cdot g) \;=\; \varphi(x, y) \; \omega_0(x, y, g) \,.$$

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$\rightsquigarrow$ Polar decomposition of φ allows to assume
$\varphi : X \times X \to$ partial isometries in $\mathcal{L}(\Lambda)$.

$\rightsquigarrow$ Let's cheat and assume $\varphi : X \times X \to \mathcal{U}(\mathcal{L}(\Lambda))$.

Sketch of the proof

So, we started with $\omega : X \times \Gamma \to \Lambda$. We defined

$$\omega_0 : \omega_0(x, y, g) = \omega(x, g)$$
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We have found that

$\omega_{1/n} \sim \omega_0$ as 1-cocycles for $\Gamma \curvearrowright X \times X$ with values in $\mathcal{U}(\mathcal{L}(\Lambda))$.

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- Applying $\alpha_{1/n}$, we obtain : $\omega_{2/n} \sim \omega_{1/n}$, ..., $\omega_1 \sim \omega_{(n-1)/n}$.
- But then, $\omega_1 \sim \omega_0$.

Sketch of the proof

Since $\omega_1 \sim \omega_0$ and

$$\omega_1(x, y, g) = \omega(y, g) \ , \ \ \omega_0(x, y, g) = \omega(x, g) \ ,$$

there exists $\varphi : X \times X \to \mathcal{U}(\mathcal{L}(\Lambda))$ with

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- Let $\varphi_0 : X \to \mathcal{U}(\mathcal{L}(\Lambda))$ be an ess. value of $\varphi : X \to \mathcal{U}(X \to \mathcal{L}(\Lambda))$.

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- Let $\varphi_0 : X \to \mathcal{U}(\mathcal{L}(\Lambda))$ be an ess. value of $\varphi : X \to \mathcal{U}(X \to \mathcal{L}(\Lambda))$.
- Then, by weak mixing,
  $$\varphi(x)^{-1}\, \omega(x, g)\, \varphi(x \cdot g) \;=\; \pi(g) \quad \text{for} \quad \pi : \Gamma \to \mathcal{U}(\mathcal{L}(\Lambda)).$$

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- We may assume that 1 is an essential value of φ.
  Again by weak mixing, $\varphi(x), \pi(g) \in \Lambda$!

End of the proof.

What did we really use

We have $\Gamma \curvearrowright (X, \mu)$ and $H \lhd \Gamma$ infinite normal subgroup with the relative property (T).

- Malleability of $\Gamma \curvearrowright (X, \mu)$.
- Weak mixing of $H \curvearrowright (X, \mu)$.
- All 1-cocycles
  with values in a closed subgroup of the unitary group of (M, τ),
  are cohomologous to a homomorphism.

A first application of cocycle superrigidity

Take $\Gamma \curvearrowright (X, \mu) = \prod_{\Gamma/\Gamma_0} (X_0, \mu_0)$. Assume

- Commensurator of $\Gamma_0 \subset \Gamma$ equals Γ_0.
- Γ has no finite normal subgroups.
- $H \lhd \Gamma$ has relative (T) with $H\Gamma_0/\Gamma_0$ infinite.

Corollary to Popa's cocycle superrigidity

The action $\Gamma \curvearrowright (X, \mu)$ is orbitally superrigid.

The orbit equivalence relation remembers $\Gamma_0 \subset \Gamma$ and (X_0, μ_0).

What we are after

- Distinguish group actions up to orbit equivalence.
- Distinguish group actions up to von Neumann equivalence : $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$.
- Even distinguish group actions
  'up to commensurablity of their von Neumann algebras'.

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⤳ II_1 factor : tracial vNalg (M, τ) having trivial center.

⤳ Distinguishing II_1 factors is an extremely hard problem.

⤳ Orbit equivalence = von Neumann equivalence
+ control of Cartan.

Group measure space construction

Let $\Gamma \curvearrowright (X, \mu)$, probability measure preserving, free, ergodic.

The II_1 factor $L^\infty(X) \rtimes \Gamma$

- contains a copy of $L^\infty(X)$,
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in such a way that

- $u_g F(\cdot) u_g^* = F(\cdot g)$,
- $\tau(F u_g) = \begin{cases} \int_X F \, d\mu & \text{if } g = e \,, \\ 0 & \text{if } g \neq e \,. \end{cases}$

Popa's von Neumann strong rigidity theorem

w-rigid group : admitting an infinite normal subgroup with the relative property (T).

Theorem (Popa, 2005)

Let Γ be *w*-rigid and ICC. Take $\Gamma \curvearrowright (X, \mu)$ free ergodic.

Let Λ be ICC and $\Lambda \curvearrowright (X_0, \mu_0)^\Lambda$ plain Bernoulli action.

If both actions are von Neumann equivalent, the groups are isomorphic and the actions conjugate.

$\rightsquigarrow$ To get hold of the Cartan subalgebras,
an extremely fine analysis is needed.

Good generalized Bernoulli actions

We study $\Gamma \curvearrowright I = \Gamma/\Gamma_0$ and $\Gamma \curvearrowright \prod_I (X_0, \mu_0)$ satisfying

- Commensurator of Γ_0 in Γ equals Γ_0.
- $H < \Gamma$ almost normal, with the relative property (T) and the relative ICC property.
- No infinite sequence (i_n) in I with $\text{Stab}(i_1, \ldots, i_n)$ **strictly decreasing**.
- For every $g \in \Gamma - \{e\}$, Fix $g \subset I$ has **infinite index**.

Good generalized Bernoulli actions

Some examples

- $\mathrm{PSL}(n,\mathbb{Z}) \curvearrowright P(\mathbb{Q}^n)$ and $\mathrm{PSL}(n,\mathbb{Q}) \curvearrowright P(\mathbb{Q}^n)$ for $n \geq 3$.
- $(\mathrm{SL}(n,\mathbb{Z}) \ltimes \mathbb{Z}^n) \curvearrowright \mathbb{Z}^n$ and $(\mathrm{SL}(n,\mathbb{Q}) \ltimes \mathbb{Q}^n) \curvearrowright \mathbb{Q}^n$ for $n \geq 2$.
- $(\Gamma \times \Gamma) \curvearrowright \Gamma$ for Γ an ICC group, with property (T), without infinite strictly decreasing sequence $C_\Gamma(g_1,\ldots,g_n)$ of centralizers.

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Write $\mathrm{vN}(\Gamma_0 \subset \Gamma, X_0, \mu_0) = L^\infty\Big(\prod_{\Gamma/\Gamma_0} (X_0, \mu_0)\Big) \rtimes \Gamma$.

Theorem (Popa-V, 2006 and V, 2007)

Under the good conditions, every isomorphism between

$$\mathrm{vN}(\Gamma_0 \subset \Gamma, X_0, \mu_0) \quad \text{and} \quad \mathrm{vN}(\Lambda_0 \subset \Lambda, Y_0, \eta_0)^t,$$

yields $t = 1$, $(\Gamma_0 \subset \Gamma) \cong (\Lambda_0 \subset \Lambda)$ and $(X_0, \mu_0) \cong (Y_0, \eta_0)$.

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Trivial Out

With $\mathrm{PSL}(n,\mathbb{Z}) \curvearrowright P(\mathbb{Q}^n)$, we get the simplest available concrete II_1 factors with trivial Out (and trivial fundamental group).

Let M be a type II_1 factor with trace τ.

- A right M-module is a Hilbert space with a right action of M.

  ⤳ Example : $L^2(M,\tau)_M$.

- Always, $H_M \cong \bigoplus_{i\in I} p_i L^2(M)$

  and one defines $\dim(H_M) = \sum_i \tau(p_i) \in [0,+\infty]$.

  ⤳ Complete invariant of right M-modules.

Definition

A bifinite M-M-bimodule, is an M-M-bimodule ${}_M H_M$ satisfying

$$\dim(H_M) < \infty \quad \text{and} \quad \dim({}_M H) < \infty.$$

The fusion algebra of bifinite bimodules

Notation : $\mathrm{FAlg}(M)$ is the set of all bifinite M-M-bimodules modulo isomorphism and called the fusion algebra of M.

$\rightsquigarrow$ Both $\mathrm{Out}(M)$ and $\mathcal{F}(M)$ are encoded in $\mathrm{FAlg}(M)$.

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The set $\mathrm{FAlg}(M)$ carries the following structure.

- Direct sum of elements in $\mathrm{FAlg}(M)$.
- Connes' tensor product $H \underset{M}{\otimes} K$ of bimodules $H, K \in \mathrm{FAlg}(M)$.
- Notion of irreducible elements.

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- Notion of irreducible elements.

$\rightsquigarrow$ FAlg(M) is a group-like invariant of II_1 factors.

$\rightsquigarrow$ We present the first explicit computations of FAlg(M).

Again generalized Bernoulli actions

Take again $\mathrm{vN}(\Gamma_0 \subset \Gamma, X_0, \mu_0) = L^\infty\Big(\prod_{\Gamma/\Gamma_0} (X_0, \mu_0)\Big) \rtimes \Gamma.$

Theorem (V, 2007)

Under the good conditions, every bifinite bimodule between

$$\mathrm{vN}(\Gamma_0 \subset \Gamma, X_0, \mu_0) \quad \text{and} \quad \mathrm{vN}(\Lambda_0 \subset \Lambda, Y_0, \eta_0)$$

is described through

- a commensurability of $\Gamma \curvearrowright \Gamma/\Gamma_0$ and $\Lambda \curvearrowright \Lambda/\Lambda_0$,
- a finite-dimensional unitary rep. of $\Gamma_1 < \Gamma$.

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⤳ General principle.
Conclusion holds whenever $\Gamma \curvearrowright (X, \mu)$ is cocycle superrigid and the bimodule 'preserves the Cartan subalgebra'.

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- a finite-dimensional unitary rep. of $\Gamma_1 < \Gamma$.

Example : trivial fusion algebra

With $(SL(2,\mathbb{Q}) \ltimes \mathbb{Q}^2) \curvearrowright \mathbb{Q}^2$ (and a scalar 2-cocycle), we get the first concrete II_1 factors without non-trivial bifinite bimodules.