Some open problems on countable Borel equivalence relations

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19th March 2007

Countable Borel equivalence relations

Definition

The Borel equivalence relation E on the standard Borel space X is said to be countable iff every E-class is countable.

Standard Example

Let G be a countable (discrete) group and let X be a standard Borel G-space. Then the corresponding orbit equivalence relation E_G^X is a countable Borel equivalence relation.

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Standard Example

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Theorem (Feldman-Moore)

If E is a countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

The space of torsion-free abelian groups of rank n

Definition

The standard Borel space of torsion-free abelian groups of rank n is defined to be

$$R(\mathbb{Q}^n) = \{ A \leqslant \mathbb{Q}^n \mid A \text{ contains a basis } \}.$$

Remark

Notice that if $A, B \in R(\mathbb{Q}^n)$, *then*

$$A \cong B \quad \text{iff} \quad \text{there exists } \varphi \in GL_n(\mathbb{Q}) \text{ such that } \varphi[A] = B.$$

Thus the isomorphism relation $\cong_n$ *on* $R(\mathbb{Q}^n)$ *is a countable Borel equivalence relation.*

The Polish space of f.g. groups

Let $\mathbb{F}_m$ be the free group on $\{x_1, \cdots, x_m\}$ and let $\mathcal{G}_m$ be the compact space of normal subgroups of $\mathbb{F}_m$. Since each m-generator group can be realised as a quotient $\mathbb{F}_m/N$ for some $N \in \mathcal{G}_m$, we can regard $\mathcal{G}_m$ as the space of m-generator groups. There are natural embeddings

$$\mathcal{G}_1 \hookrightarrow \mathcal{G}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{G}_m \hookrightarrow \cdots$$

and we can regard

$$\mathcal{G} = \bigcup_{m \geq 1} \mathcal{G}_m$$

as the space of f.g. groups.

Theorem (Champetier)

The isomorphism relation $\cong$ on the space $\mathcal{G}$ of f.g. groups is a countable Borel equivalence relation.

Borel reductions

Definition

Let E, F be Borel equivalence relations on the standard Borel spaces X, Y respectively.

- $E \leq_B F$ *iff there exists a Borel map* $f : X \to Y$ *such that*

$$x \, E \, y \iff f(x) \, F \, f(y).$$

  In this case, f *is called a* Borel reduction *from* E *to* F.
- $E \sim_B F$ *iff both* $E \leq_B F$ *and* $F \leq_B E$.
- $E <_B F$ *iff both* $E \leq_B F$ *and* $E \nsim_B F$.

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Definition

More generally, $f : X \to Y$ *is a* **Borel homomorphism** *from E to F iff*

$$x \, E \, y \implies f(x) \, F \, f(y).$$

A Cantor-Bernstein Theorem

Theorem

If E, F are countable Borel equivalence relations on the standard Borel spaces X, Y, then the following are equivalent:

- $E \sim_B F$.
- *There exist complete Borel sections $A \subseteq X$ and $B \subseteq Y$ such that*

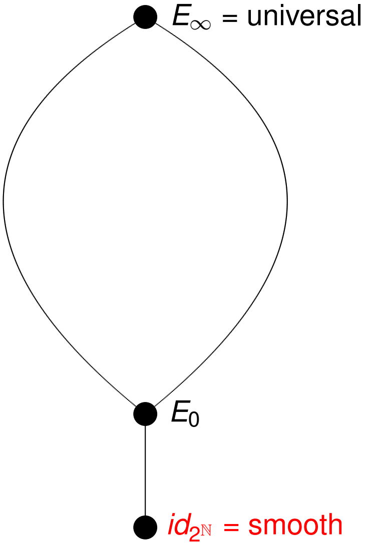
$$(A, E \restriction A) \cong (B, F \restriction B)$$

  via a Borel isomorphism.

Definition

A Borel subset $A \subseteq X$ is a complete section iff A intersects every E-class.

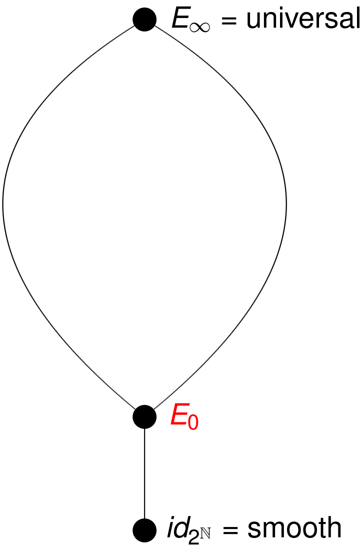
Countable Borel equivalence relations



Definition

The Borel equivalence relation E is smooth iff $E \leq_B id_{2^{\mathbb{N}}}$, where $2^{\mathbb{N}}$ is the space of infinite binary sequences.

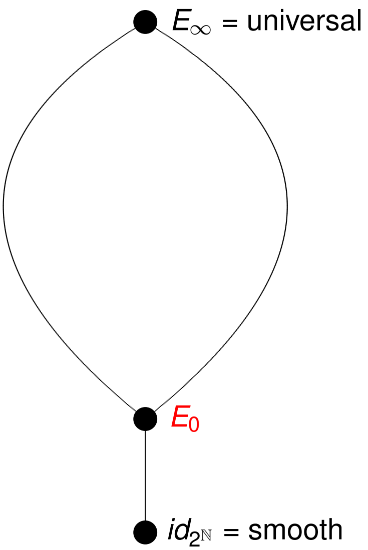
Countable Borel equivalence relations



Definition

E_0 is the equivalence relation of *eventual equality* *on the space $2^{\mathbb{N}}$ of infinite binary sequences.*

Countable Borel equivalence relations



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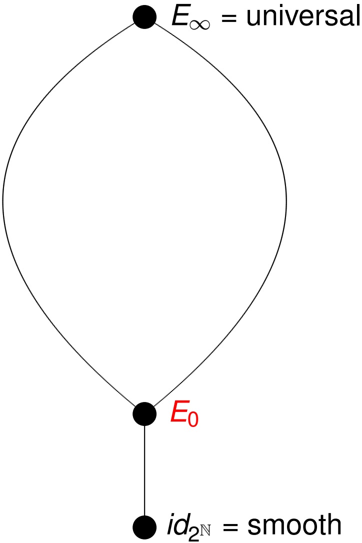
Question

Does there exist a nonsmooth countable Borel E with an immediate $<_B$-successor?

Countable Borel equivalence relations



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Does there exist a nonsmooth countable Borel E with no immediate $<_B$-successor?

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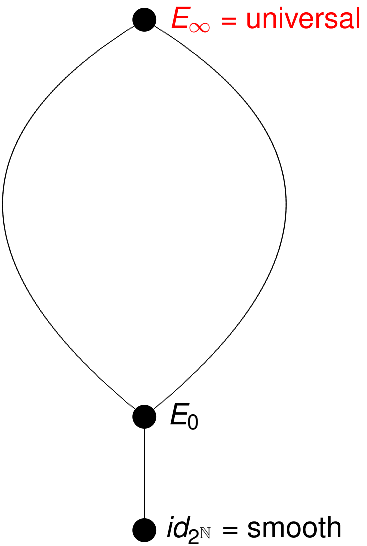
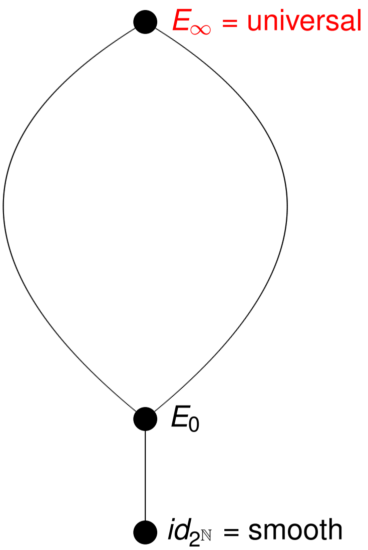


Definition

A countable Borel equivalence relation E is universal iff $F \leq_B E$ for every countable Borel equivalence relation F.

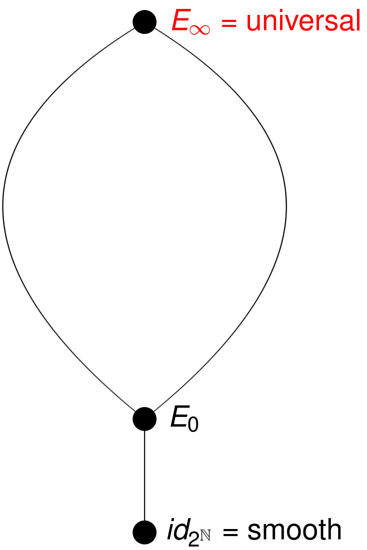
Countable Borel equivalence relations



Theorem (JKL)

The orbit equivalence relation E_∞ of the action of the free group $\mathbb{F}_2$ on its powerset $\mathcal{P}(\mathbb{F}_2) = 2^{\mathbb{F}_2}$ is countable universal.

Countable Borel equivalence relations



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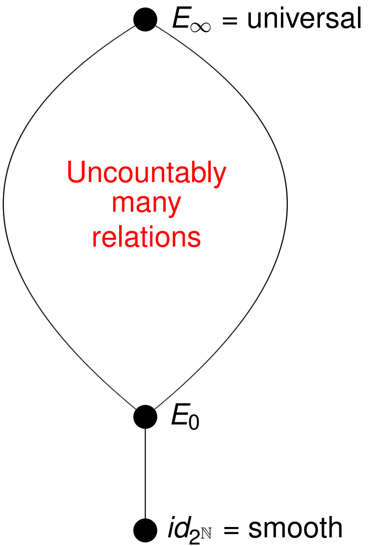
Theorem (TV)

The isomorphism relation on the space of f.g. groups is countable universal.

Countable Borel equivalence relations



Theorem (Adams-Kechris 2000)

There exist $2^{\aleph_0}$ many countable Borel equivalence relations up to Borel bireducibility.

The measurable vs. Borel settings

Let G be a countable group and let X be a standard Borel G-space.

The Fundamental Question in the Borel setting

To what extent does the data (X, E_G^X) "remember" G and its action on X?

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Fact

We cannot possibly recover the group G from the data (X, E_G^X) unless we add the hypotheses that:

- *G acts freely on X.*
- *there exists a G-invariant probability measure μ on X.*

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Let E be a nonsmooth countable Borel equivalence relation. Does there necessarily exist a countable group G with a free measure-preserving Borel action on a standard probability space (X, μ) such that $E \sim_B E_G^X$?

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Definition

- *The countable Borel equivalence relation E on X is free iff there exists a countable group G with a free Borel action on X such that $E_G^X = E$.*
- *The countable Borel equivalence relation E is essentially free iff there exists a free countable Borel equivalence relation F such that $E \sim_B F$.*

Some closure properties

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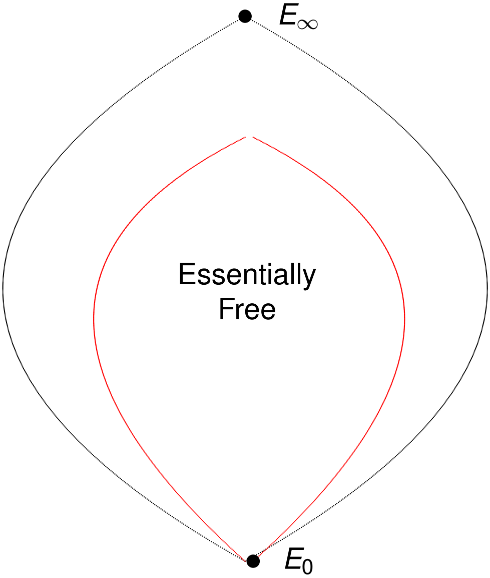
- *If $E \leq_B F$ and F is essentially free, then so is E.*
- *If $E \subseteq F$ and F is essentially free, then so is E.*

Corollary

The following statements are equivalent:

- *Every countable Borel equivalence relation is essentially free.*
- *E_∞ is essentially free.*

Essentially free countable Borel equivalence relations



Theorem (S.T.)

The class of essentially free countable Borel equivalence relations does not admit a universal element.

Corollary

E_∞ is not essentially free.

Bernoulli actions

- Let G be a countably infinite group and let μ be the usual product probability measure on $\mathcal{P}(G) = 2^G$.

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Observation

If $G \leqslant H$, then $E_G \leq_B E_H$.

Proof.

The inclusion map $\mathcal{P}^*(G) \hookrightarrow \mathcal{P}^*(H)$ is a Borel reduction from E_G to E_H. □

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Definition

If G, H are countable groups, then the group homomorphism $\pi : G \to H$ is a virtual embedding iff $|\ker \pi| < \infty$.

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If there exists a μ-nontrivial Borel homomorphism from E_G to E_H^Y, then there exists a virtual embedding $\pi : G \to H$.

Corollary

If S, T are countable groups with no nontrivial finite normal subgroups, then the following are equivalent:

- $E_{SL_3(\mathbb{Z}) \times S} \leq_B E_{SL_3(\mathbb{Z}) \times T}$.
- *$SL_3(\mathbb{Z}) \times S$ embeds into $SL_3(\mathbb{Z}) \times T$.*

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- We can suppose that $E = E_H^X$ is realised by a free Borel action on X of the countable group H.

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- Let L be a f.g. group which does not embed into H.

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- Let $S = L * \mathbb{Z}$ and let $G = SL_3(\mathbb{Z}) \times S$.

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- Let $S = L * \mathbb{Z}$ and let $G = SL_3(\mathbb{Z}) \times S$.
- Then G has no finite normal subgroups and so there does not exist a virtual embedding $\pi : G \to H$.
- Hence $E_G \not\leq_B E_H^X$.

□

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Lemma

There exists a Borel family $\{S_x \mid x \in 2^{\mathbb{N}}\}$ of f.g. groups such that if $G_x = SL_3(\mathbb{Z}) \times S_x$, then the following conditions hold:

- *If $x \neq y$, then G_x and G_y are not isomorphic up to finite kernels.*
- *If $x \neq y$, then G_x doesn't virtually embed in G_y.*

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Definition

For each Borel subset $A \subseteq 2^{\mathbb{N}}$, let $E_A = \bigsqcup_{x \in A} E_{G_x}$ on $\bigsqcup_{x \in A} (2)^{G_x}$.

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Lemma

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- Suppose that $E_A \leq_B E_H^Y$, where H is a countable group and Y is a free standard Borel H-space.
- Then for each $x \in A$, we have that $E_{G_x} \leq_B E_H^Y$ and so there exists a virtual embedding $\pi_x : G_x \to H$.

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- Since A is uncountable, there exist $x \neq y \in A$ such that $\pi_x[G_x] = \pi_y[G_y]$.
- But then G_x, G_y are isomorphic up to finite kernels, which is a contradiction.

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- This yields a μ_x-nontrivial Borel homomorphism from E_{G_x} to E_{G_y} and so G_x virtually embeds into G_y, which is a contradiction.

Smooth disjoint unions

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Suppose that $E_\infty = \bigsqcup_{z \in A} E_z$ is expressed as a smooth disjoint union of countable Borel equivalence relations $\{E_z \mid z \in A\}$. Does there necessarily exist an element $z \in A$ such that E_z is countable universal?

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Remark

The previous question remains open when $A = \{1, 2\}$.

Partitions of the space of f.g. groups

Recall that the isomorphism relation $\cong$ on the standard Borel space $\mathcal{G}$ of f.g. groups is countable universal.

Question

Suppose $\mathcal{G}$ is partitioned into two $\cong$-invariant Borel subsets

$$\mathcal{G} = X \sqcup Y.$$

Is it necessarily the case that either $\cong \restriction X$ or $\cong \restriction Y$ countable universal?

Strongly universal relations

Definition

Suppose that E is a countable Borel equivalence relation on the standard Borel space X with invariant ergodic probability measure μ. Then E is strongly universal iff $E \restriction A$ is universal for every Borel subset $A \subseteq X$ with $\mu(A) = 1$.

Question

Does there exist a strongly universal countable Borel equivalence relation?

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Question

Suppose that E is a countable Borel equivalence relation on the standard Borel space X with invariant ergodic probability measure μ. Does there always exist a Borel subset $A \subseteq X$ with $\mu(A) = 1$ such that $E \restriction A$ is essentially free?

Equivalently ...

Question

Suppose that E is a countable Borel equivalence relation on the standard Borel space X with invariant ergodic probability measure μ. Does there always exist a Borel subset $A \subseteq X$ with $\mu(A) > 0$ such that $(E \restriction A) \times I(\mathbb{N})$ is free?

Definition

Here $I(\mathbb{N})$ is the equivalence relation on $\mathbb{N}$ such that all points are equivalent.