The free group factors conundrum

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Sorin Popa

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• This problem gave rise, directly or indirectly, to a huge amount of mathematics, a multitude of concepts and insightful techniques: approximation properties (compact, weak amenability, the Λ-invariant), absorption/repelling/tree-like behavior of $L\mathbb{F}_n$ as bimodule over its amenable subalgebras, free probability methods (random matrix models and free entropy), deformation-rigidity/intertwining methods, W^*-boundary methods (bi-exactness and proper proximality), thin&tight decomposition methods, L^2-cohomology attempts, bounded generation, ...

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• The difficulty seems to come from the very nature of $L\mathbb{F}_n$: a subtle mixture of "mild rigidity" (due to spectral gap of $\mathbb{F}_n \curvearrowright L\mathbb{F}_n$ and "tree-ness") with a multitude of deformation properties (free malleability, compact c.p., finite rank c.b.). With both features "spread out" inside $L\mathbb{F}_n$ in a random manner!

Some key open problems about $L\mathbb{F}_n$

Given a II_1 factor M, I'll denote by $\text{ng}(M)$ the minimal number $2 \leq n \leq \infty$ of selfadjoint elements that can generate M as a vN algebra.

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(4) The *freely complemented* (FC) *problem*: Is any maximal amenable $B \subset L\mathbb{F}_n$ FC in $L\mathbb{F}_n$, i.e. $\exists N$ s.t. $L\mathbb{F}_n = B * N$? Notably for B a MASA.

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(5) *vN-type problem*: $L\mathbb{F}_n \hookrightarrow M$ for any non-amenable II_1 factor M?

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• Peterson-Thom conjecture (2011): if $Q \subset M = L\mathbb{F}_n$ is maximal amenable then any $Q_0 \subset L\mathbb{F}_n$ amenable with $Q_0 \cap Q$ diffuse must be contained Q. Strengthened conjecture by Hayes, Popa (2019): any maximal amenable $Q \subset L\mathbb{F}_n$ is coarse.

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• One hitch about all this: for any freely complemented $Q \subset L\mathbb{F}_n$ one already knows both PT and coarseness hold true (Popa 82), and there are no known examples of maximal amenable $Q \subset L\mathbb{F}_n$ that are not FC!

Test cases for the FC problem

Recall that for any B diffuse amenable, $B \subset M = B * N$ is maximal amenable, PT-absorbing and coarse.

(*a*) Let $g \in \mathbb{F}_n$ be so that $g^{\mathbb{Z}}$ is maximal abelian in $\mathbb{F}_n$. Then $A_g = \{u_g\}''$ is maximal amenable in $L\mathbb{F}_n$ (P1982). Is A_g freely complemented in $L\mathbb{F}_n$, even if $g^{\mathbb{Z}}$ is not freely complemented in $\mathbb{F}_n$?

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(*b*) Is the *radial MASA* $L_n \subset L\mathbb{F}_n$, $2 \leq n < \infty$, defined by $L_n = \{\sum_{i=1}^{n}(u_i + u_i^*)\}''$ freely complemented? (NB: L_n is known to be maximal amenable by Cameron-Fang-Ravichandran-White 2010).

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(*c*) the Boutonnet-Popa examples (2022): Let $\{(M_j, \tau_j)\}_{j \in J}$ be tracial vN algebras, with $s_j \in M_j$ semicircular, $\forall j$. Denote ℓ^2_* the set of square summable J-tuples/$\mathbb{R}$ with at least two non-zero entries. For each $t = (t_j)_j \in \ell^2_*$ denote by $A(t)$ the abelian vN generated in $M = *_{j \in J} M_j$ by $s(t) := \sum_j t_j s_j \in M$. Then $A(t)$ is maximal amenable in M, $\forall t \in \ell^2_*$, with $A(t) \prec_M A(t')$ iff $t, t' \in \ell^2_*$ proportional.

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(*d*) If $\{B_i\}_i$ are diffuse amenable vN in $M = L\mathbb{F}_n$ with B_i freely complemented and $B_i \not\prec_{L\mathbb{F}_n} B_j$, $\forall i \neq j$, then $B = \oplus_i u_i p_i B_i p_i u_i^*$ is maximal amenable in M for any $p_i \in \mathcal{P}(B_i)$ and $u_i \in \mathcal{U}(M)$ satisfying

weak FC conjectures

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The weak FC conjectures

• Given any amenable $B \subset M = L\mathbb{F}_n$ there exists a Haar unitary $u \in M$ that's free independent to B.
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• As we will hear in the Wed talk, Boschert-Davis-Hiatt have shown first conj. holds true in the above examples $(a), (b), (c)$.

Non-iso of A^{*n} for non-separable A

• The B-P examples and the proofs involved naturally led to the problem of whether any purely non-separable (singular) MASA B in $M = A^{*n}$, with A purely non-separable is "made up" of pieces of $A_k := 1 * ... * A * ...1$ (kth position), a fact that would imply that n is "remembered" by the iso-class of A^{*n}! Indeed one has:

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Theorem (Boutonnet–Drimbe-Ioana-Popa 03/2023)

Let A be a non-separable tracial vN algebra. Then $A^{*n}, 2 \leq n \leq \infty$, are mutually non-isomorphic, with $\mathcal{F}(A^{*n}) = 1$ whenever $n < \infty$.

The "classic" free group factor problems (1), (2), (3)

• It doesn't seem possible to use the non-iso of A^{*n} for non-separable A to deduce the non-iso of the $L\mathbb{F}_n$. But note hat if $M = A^{*n}$ with A purely ns, then $\exists N_i \nearrow M$ subfactors such that $N_i \simeq L\mathbb{F}_n$, $\forall i$.

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• By results of Dykema, Radulescu (1992) using Voiculescu's free probability, if $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$ (so if (3) holds true) then $L\mathbb{F}_n$, $2 \leq n \leq \infty$, non-iso and $\mathcal{F}(L\mathbb{F}_n) = 1$, $\forall n < \infty$. So (1) and (2) would follow as well.

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- *Tightness conjecture* states that if a II_1 factor M has stably bounded number of generators, i.e. $\sup_t \mathrm{ng}(M^t) < \infty$, then M is *R-tight*: $\exists R_0, R_1 \subset M$ such that ${}_{R_0}L^2M_{R_1}$ is irreducible. In particular, if M is finitely generated and $\mathcal{F}(M) \neq 1$, then M would follow R-tight. Since $\mathcal{F}(L\mathbb{F}_\infty) \neq 1$ (Voiculescu 1988, Radulescu 1991), this would show that if $\mathrm{ng}(L\mathbb{F}_\infty) < \infty$ then $L\mathbb{F}_\infty$ is tight, contradicting Ge-Popa 1996.

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• Thus, tightness conjecture implies $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$, more generally $\exists 1 \geq c > 0$ such that $n \geq \mathrm{ng}(L\mathbb{F}_n) \geq cn$, $\forall 2 \leq n \leq \infty$. So by the remarks above, the tightness conjecture solves (1), (2), (3).

Does any non-amenable II_1 factor contain $L\mathbb{F}_2$?

• Recall that by Olshanski (1980) there exist non-amenable groups Γ such that: (1) any $g \in \Gamma$ has torsion; (2) if $h \in \Gamma$ is not a power of g, then g, h generate the whole group Γ. Thus, such Γ cannot contain $\mathbb{F}_2$, answering in the negative the "classic vN problem". But its II_1 factor version (5) remains open.

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- An obvious "test case" is the group factor $L\Gamma$ with the Γ as above. Other examples to try are crossed products by Γ, such as $M = L^\infty(X, \mu) \rtimes \Gamma$, or $M = R \rtimes \Gamma$, where $\Gamma \curvearrowright (X, \mu)$ free ergodic p.m.p. and $\Gamma \curvearrowright R$ free.

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• Note that in case Γ acts on (X, μ) or on R by Bernoulli shifts, then the corresponding M does contain $L\mathbb{F}_2$ (Gaboriau-Lyons 2011).

A side remark

• Note that given any Γ and any free action $\Gamma \curvearrowright R$, there is a Galois correspondence between subgroups $H \subset \Gamma$ and intermediate subfactors $R \subset N \subset M = R \rtimes \Gamma$ (Choda 78). In particular, between maximal amenable subgroup $H \subset \Gamma$ and maximal amenable subfactors $N \subset M$ that contain R, $\Gamma \supset H \mapsto N_H \subset M$, $N \mapsto H_N \subset \Gamma$.

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• Given any $n \geq 1$, an appropriate choice of $H_0 \subset \Gamma$ with H_0 amenable, gives example $R_0 = R \rtimes H_0 \subset R \rtimes \Gamma = M$ of hyperfinite subfactors $R_0 \subset M$ with exactly n maximal amenable $R_0 \subset N \subset M$.

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• In case Γ is the Olsanski group, any maximal amenable $H_0 \subset \Gamma$ is a finite cyclic group and the corresponding maximal amenable $R \rtimes H_0 = N \subset M = R \times \Gamma$ is quasi-regular in M, and is in fact "super-maximal", in that there are no proper subfactors betweenn N and M.

A class of "small" non-amenable II_1 factors [P94]

• Let $\mathcal{C}$ be a non-degenerate commuting square of tracial multi-matrix algebras $(P_{00} \subset P_{01}) \subset (P_{10} \subset P_{11})$ with all inclusion bipartite graphs irreducible. By iterating the basic construction "horizontally" one gets a sequence of commuting squares with the limit $P_{0\infty} \subset P_{1\infty}$ being a hyperfinite subfactor with Jones index equal to $\|G\|^2$, where G is the bipartite graph of the "initial" vertical inclusion $P_{00} \subset P_{10}$.

Let $T(\mathcal{C}) \subset S(\mathcal{C})$ be the *symmetric enveloping inclusion* of II_1 factors associated with this subfactor (as defined in [P94]: describe!). Then

• $T \simeq R \otimes R^{op}$.
• $T \subset S$ is quasi-regular (crossed-product type inclusion).
• If $4 < \|G\|^2 < 2 + \sqrt{5}$ then: S is non-amenable, $T \subset S$ has no intermediate subfactors (in particular T is maximal amenable in S). Also, S has Haagerup property relative to T ([PV2016]).
• The underlying C*-algebra $C^*(P, e_N, P^{op}) \subset \mathcal{B}(L^2 P)$ is quasi-diagonal, where $P = \cup_n P_{1n}$.