Are there any maximal amenable subalgebras in $L\mathbb{F}_n$ that are not freely complemented?

UCSD, February 11, 2023

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• P 82: Q diffuse amenable implies Q is maximal amenable in $M = Q * M_2$. Proof also shows that: (a) If $Q_0 \subset M = Q * M_2$ amenable with $Q_0 \cap Q$ diffuse then $Q_0 \subset Q$ (PT-absorption); (b) If $Q \subset M_1$ maximal amenable then $Q \subset M_1 * M_2$ maximal amenable (amenable absorption). Generalized to $M_1 *_B M_2$ Boutonnet-Hudayer 16

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- Voiculescu free probability 91-94: Structural randomness in $L\mathbb{F}_n$, quantified as *free entropy*, prevents "thinness around MASAs": $L\mathbb{F}_n$ have no Cartan (V94); $L\mathbb{F}_n$ is prime (Ge 96); $L\mathbb{F}_n$ is not thin around arbitrary AFD (Ge-Popa 96).

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- P 01, Ozawa 03, P 06, Peterson 06, Ozawa-P 07, Ding-Kunnawalkam-Peterson 22: *deformation-rigidity* and *boundary methods* prevent non-amenable Q with "too much structure" from being embeddable into $L\mathbb{F}_n$.

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The problem

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Maximal amenable radial-like MASAs

Theorem (Boutonnet-Popa 2022)

Let $\{(M_j, \tau_j)\}_{j \in J}$ be tracial vN algebras, with $s_j \in M_j$ semicircular, $\forall j$. Denote ℓ^2_* the set of square summable J-tuples$/\mathbb{R}$ with at least two non-zero entries. For each $t = (t_j)_j \in \ell^2_*$ denote by $A(t)$ the abelian vN generated in $M = *_{j \in J} M_j$ by $s(t) := \sum_j t_j s_j \in M$. Then $A(t)$ is maximal amenable in M, $\forall t \in \ell^2_*$, with $A(t) \prec_M A(t')$ iff $t, t' \in \ell^2_*$ proportional.

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Proof uses the II_1 subfactor $N \subset M$ generated by $\{s_j\}_j \subset M$. Note that $N \simeq L\mathbb{F}_J$.

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If $H = H(J)$ denotes the J-dim Hilbert space/$\mathbb{R}$, we alternatively view N as the vN generated by $s(\xi)$, $\xi \in H$, on $\mathcal{F}(H)$, where $\mathcal{F}(H)$ is the full Fock space of H and $s(\xi) \in N$ is the semi-circular operator associated to the unit vector ξ in H (Voiculescu).

Non-conjugacy of the $A(\xi)$ in N

We first establish the non-conjugacy of $A(\xi)$ in N, $\xi \in H$, where N is viewed via the above free Gaussian functor:

Fact 1

Given unit vectors $\xi_1, \xi_2 \in H$, we have $A(\xi_1) \prec_N A(\xi_2)$ iff $A(\xi_1) = A(\xi_2)$ iff $\xi_1 = \pm\xi_2$.

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Recall: $A(\xi) \prec_N A(\eta)$ iff $\exists v \in N, \neq 0$, partial iso such that $v^*v \in A(\xi)$, $vv^* \in A(\eta)$ and $vA(\xi)v^* = A(\eta)vv^*$ (P 2001).

Also: if $\xi \perp \eta$ then $A(\xi), A(\eta)$ free in N, so $A(\xi) \not\prec_N A(\eta)$ (P81).

Now note that for unit vectors $\xi, \eta, \eta' \in H$ with $\langle \xi, \eta \rangle = \langle \xi, \eta' \rangle$, $\exists \alpha \in \mathcal{O}(H)$ such that $\alpha(\xi) = \xi$ and $\alpha(\eta) = \eta'$. The associated automorphism $\theta = \theta_\alpha$ of $N = L\mathbb{F}_J$ leaves $A(\xi)$ fixed and $\theta(A(\eta)) = A(\eta')$. Thus, if $v \in N$ is a partial isometry with $v^*v \in A(\xi)$, $vv^* \in A(\eta)$ and $vA(\xi)v^* = A(\eta)vv^*$, then

$$\theta(v)A(\xi)\theta(v)^* = \theta(vA(\xi)v^*) = \theta(A(\eta)vv^*) = A(\eta')\theta(vv^*),$$

implying that $w = \theta(v)v^* \in N$ is a partial iso with $w^*w = vv^* \in A(\eta)$, $ww^* = \theta(vv^*) \in A(\eta')$ and satisfying $wA(\eta)w^* = A(\eta')ww^*$.

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Applying this observation appropriately several times, starting from ξ_1, ξ_2 with $A(\xi_1) \prec_N A(\xi_2)$ one can get unit vectors $\xi, \eta \in H$ such that $\xi \perp \eta$ and $A(\xi) \prec_N A(\eta)$, contradiction.

Background on absorption

Fact 2 (Ioana-Peterson-Popa 2005)

Let $B \subset P_1, B \subset P_2$ be inclusions of tracial von Neumann algebras, with $\tau_{P_1|B} = \tau_{P_2|B}$, and let $P = P_1 *_B P_2$. If $A_1 \subset P_1$ is vN such that $A_1 \not\prec_{P_1} B$, then for all $A_2 \subset P_1$ one has $A_1 \not\prec_P A_2$ iff $A_1 \not\prec_{P_1} A_2$.

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Fact 3 (Popa 1982, Boutonnet-Houdayer 2016)

If Q amenable diffuse, then Q is maximal amenable in $Q * M_2$. More generally, if $Q \subset M_1$ is maximal amenable in M_1 and $Q \not\prec_{M_1} B$, then Q is maximal amenable in $M = M_1 *_B M_2$.

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Fact 4 (folklore)

Let $B \subset P$, and Q be tracial vN. Then $(B * Q \subset P * Q) \simeq (B * Q \subset P *_B (B * Q))$.

Proof of the Theorem

Sufficient to prove $J = \{1, 2, ..., n\}$ finite. Denote $N_0 = N$ and for $i \geq 1$ let $N_i = M_1 * ... * M_i * A(s_{i+1}) * ... * A(s_n)$. Note that by Fact 4 we have $(N_i \subset N_{i+1}) = (N_i \subset N_i *_{A(s_i)} M_i)$.

Applying Facts 1, 2, 3 and using the above identification of $N_i \subset N_{i+1}$, we obtain recursively in $i \geq 0$ that $A(t) \not\prec_{N_i} A(\xi)$, if t, ξ are not colinear, in particular for $\xi = s_j$, $\forall j$, and that $A(\xi)$ is maximal amenable in N_i, for each $i = 0, 1, 2, ..., n$. Since $N_n = M$, we are done.

Interesting cases to investigate

- The case $M_j = A(s_j) \otimes R$, $\forall j \in J$, where $A(s_j) = \{s_j\}'' \subset M_j$.
- $M_j = A(s_j) \rtimes \Gamma_j$, where Γ_j is an amenable group and $\Gamma_j \curvearrowright A_j$ is a trace preserving action, $\forall j \in J$.
- M_j abelian non-separable, e.g., an ultrapower of $L^\infty[0,1]$, $\forall j$.

Some related problems

• Let $g \in \mathbb{F}_n$ be so that $g^{\mathbb{Z}}$ is maximal abelian in $\mathbb{F}_n$. Then $A_g = \{u_g\}''$ is maximal amenable in $L\mathbb{F}_n$ (Popa 1982). Is A_g freely complemented in $L\mathbb{F}_n$, even if $g^{\mathbb{Z}}$ is not freely complemented in $\mathbb{F}_n$?

• Is the radial MASA $L_n \subset L\mathbb{F}_n$, $2 \leq n < \infty$, defined by $L_n = \{\sum_{i=1}^{n}(u_i + u_i^*)\}''$ freely complemented? (NB: L_n is known to be maximal amenable by Cameron-Fang-Ravichandran-White 2010).

• Does any amenable vN subalgebra $B \subset L\mathbb{F}_n$ admit Haar unitaries $u \in L\mathbb{F}_n$ that are free independent to B.

• Let A be an abelian non-separable tracial vN, e.g. an ultrapower of $L^\infty[0,1]$. Are the II_1 factors A^{*n}, $n = 2, 3, \ldots$, mutually non-isomorphic?