Some remarks on the free group factors

IMJ Paris, September 2023

Sorin Popa

The free group factors conundrum

- The II_1 factors $L\mathbb{F}_n$, arising from the free groups with n generators, $2 \leq n \leq \infty$, emerged as a fundamental class of objects in non-commutative analysis (aka operator algebras).

The free group factors conundrum

• The II_1 factors $L\mathbb{F}_n$, arising from the free groups with n generators, $2 \leq n \leq \infty$, emerged as a fundamental class of objects in non-commutative analysis (aka operator algebras).

• Despite recent progress in classifying large classes of II_1 factors arising from various data (such as groups and their action on spaces), a large number of basic questions concerning the free group factors remained open. Most notably: are the $L(\mathbb{F}_n)$, $2 \leq n \leq \infty$, non-isomorphic?

The free group factors conundrum

• The II_1 factors $L\mathbb{F}_n$, arising from the free groups with n generators, $2 \leq n \leq \infty$, emerged as a fundamental class of objects in non-commutative analysis (aka operator algebras).

• Despite recent progress in classifying large classes of II_1 factors arising from various data (such as groups and their action on spaces), a large number of basic questions concerning the free group factors remained open. Most notably: are the $L(\mathbb{F}_n)$, $2 \leq n \leq \infty$, non-isomorphic?

• This problem gave rise, directly or indirectly, to a huge amount of mathematics, a multitude of concepts and insightful techniques: approximation properties (compact, weak amenability, the Λ-invariant), absorption/repelling/branchy behavior of $L\mathbb{F}_n$ as bimodule over its amenable subalgebras, free probability methods (random matrix models and free entropy), deformation/rigidity/intertwining methods, W^*-boundary methods (bi-exactness and proper proximality), thin&tight decomposition methods, L^2-cohomology attempts, bounded generation, ...

The free group factors conundrum

• It is evident by now that the structure and classification of the free group factors $L\mathbb{F}_n$ involves some extremely fine, elusive phenomenology, still far from being understood.

The free group factors conundrum

• It is evident by now that the structure and classification of the free group factors $L\mathbb{F}_n$ involves some extremely fine, elusive phenomenology, still far from being understood.

• The difficulty seems to come from the very nature of $L\mathbb{F}_n$: a subtle mixture between "mild rigidity" (due to spectral gap of $\mathbb{F}_n \curvearrowright L\mathbb{F}_n$ and "branchyness") and a multitude of deformation properties (free malleability, compact c.p., finite rank c.b.). An information that's "spread out" inside $L\mathbb{F}_n$ in a random manner!

The free group factors conundrum

• It is evident by now that the structure and classification of the free group factors $L\mathbb{F}_n$ involves some extremely fine, elusive phenomenology, still far from being understood.

• The difficulty seems to come from the very nature of $L\mathbb{F}_n$: a subtle mixture between "mild rigidity" (due to spectral gap of $\mathbb{F}_n \curvearrowright L\mathbb{F}_n$ and "branchyness") and a multitude of deformation properties (free malleability, compact c.p., finite rank c.b.). An information that's "spread out" inside $L\mathbb{F}_n$ in a random manner!

• My perception of a free group factor $L\mathbb{F}_n$ is that it is like mud, a homogeneous mixture of earth and water without "points d'appui" !

Some key problems about $L\mathbb{F}_n$

Given a II_1 factor M, I'll denote by $\mathrm{ng}(M)$ the minimal number $2 \leq n \leq \infty$ of selfadjoint elements that can generate M as a vN algebra.

(1) The *non-isomorphism problem*: $L\mathbb{F}_n \simeq L\mathbb{F}_m$ iff $n = m$.

Some key problems about $L\mathbb{F}_n$

Given a II_1 factor M, I'll denote by $\mathrm{ng}(M)$ the minimal number $2 \leq n \leq \infty$ of selfadjoint elements that can generate M as a vN algebra.

(1) The *non-isomorphism problem*: $L\mathbb{F}_n \simeq L\mathbb{F}_m$ iff $n = m$.

(2) The *fundamental group problem*: $\mathcal{F}(L\mathbb{F}_n) = 1$ when $n < \infty$?

Some key problems about $L\mathbb{F}_n$

Given a II_1 factor M, I'll denote by $\text{ng}(M)$ the minimal number $2 \leq n \leq \infty$ of selfadjoint elements that can generate M as a vN algebra.

(1) The *non-isomorphism problem*: $L\mathbb{F}_n \simeq L\mathbb{F}_m$ iff $n = m$.

(2) The *fundamental group problem*: $\mathcal{F}(L\mathbb{F}_n) = 1$ when $n < \infty$?

(3) The finite/infinite generation problem: $\text{ng}(L\mathbb{F}_\infty) = \infty$? $\text{ng}(L\mathbb{F}_n) = n$?

Some key problems about $L\mathbb{F}_n$

Given a II_1 factor M, I'll denote by $\mathrm{ng}(M)$ the minimal number $2 \leq n \leq \infty$ of selfadjoint elements that can generate M as a vN algebra.

(1) The *non-isomorphism problem*: $L\mathbb{F}_n \simeq L\mathbb{F}_m$ iff $n = m$.

(2) The *fundamental group problem*: $\mathcal{F}(L\mathbb{F}_n) = 1$ when $n < \infty$?

(3) The finite/infinite generation problem: $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$? $\mathrm{ng}(L\mathbb{F}_n) = n$?

(4) The *freely complemented* (FC) *problem*: Is any maximal amenable $B \subset L\mathbb{F}_n$ FC in $L\mathbb{F}_n$, i.e. $\exists N$ s.t. $L\mathbb{F}_n = B * N$. Notably for B abelian.

Some key problems about $L\mathbb{F}_n$

Given a II_1 factor M, I'll denote by $\mathrm{ng}(M)$ the minimal number $2 \leq n \leq \infty$ of selfadjoint elements that can generate M as a vN algebra.

(1) The *non-isomorphism problem*: $L\mathbb{F}_n \simeq L\mathbb{F}_m$ iff $n = m$.

(2) The *fundamental group problem*: $\mathcal{F}(L\mathbb{F}_n) = 1$ when $n < \infty$?

(3) The finite/infinite generation problem: $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$? $\mathrm{ng}(L\mathbb{F}_n) = n$?

(4) The *freely complemented* (FC) *problem*: Is any maximal amenable $B \subset L\mathbb{F}_n$ FC in $L\mathbb{F}_n$, i.e. $\exists N$ s.t. $L\mathbb{F}_n = B * N$. Notably for B abelian.

(5) *vN-type problem*: $L\mathbb{F}_n \hookrightarrow M$ for any non-amenable II_1 factor M?

Some key problems about $L\mathbb{F}_n$

Given a II_1 factor M, I'll denote by $\mathrm{ng}(M)$ the minimal number $2 \leq n \leq \infty$ of selfadjoint elements that can generate M as a vN algebra.

(1) The *non-isomorphism problem*: $L\mathbb{F}_n \simeq L\mathbb{F}_m$ iff $n = m$.

(2) The *fundamental group problem*: $\mathcal{F}(L\mathbb{F}_n) = 1$ when $n < \infty$?

(3) The finite/infinite generation problem: $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$? $\mathrm{ng}(L\mathbb{F}_n) = n$?

(4) The *freely complemented* (FC) *problem*: Is any maximal amenable $B \subset L\mathbb{F}_n$ FC in $L\mathbb{F}_n$, i.e. $\exists N$ s.t. $L\mathbb{F}_n = B * N$. Notably for B abelian.

(5) *vN-type problem*: $L\mathbb{F}_n \hookrightarrow M$ for any non-amenable II_1 factor M?

(6) *Characterize factors/groups embeddable into* $L\mathbb{F}_n$. Is any II_1 factor $N \subset L\mathbb{F}_n$ iso to either R or $L\mathbb{F}_t$, $1 < t \leq \infty$? If $N \subset L\mathbb{F}_n$ subfactor with finite index then $N \simeq L\mathbb{F}_t$, where $t = 1 + [L\mathbb{F}_n : N](n-1)$.

The “free complementation” (FC) problem

- Are there maximal amenable vN subalgebras in $L\mathbb{F}_n$ that are not FC ?
Are there maximal amenable MASAs that are not FC?

The "free complementation" (FC) problem

- Are there maximal amenable vN subalgebras in $L\mathbb{F}_n$ that are not FC ?

Are there maximal amenable MASAs that are not FC?

Motivation

- Peterson-Thom conjecture (2011): if $Q \subset M = L\mathbb{F}_n$ is maximal amenable then any $Q_0 \subset L\mathbb{F}_n$ amenable with $Q_0 \cap Q$ diffuse implies must be contained Q. Strengthened conjecture by Hayes, Popa (2019): any maximal amenable $Q \subset L\mathbb{F}_n$ is coarse.

The "free complementation" (FC) problem

• Are there maximal amenable vN subalgebras in $L\mathbb{F}_n$ that are not FC ?
Are there maximal amenable MASAs that are not FC?

Motivation

• Peterson-Thom conjecture (2011): if $Q \subset M = L\mathbb{F}_n$ is maximal amenable then any $Q_0 \subset L\mathbb{F}_n$ amenable with $Q_0 \cap Q$ diffuse implies must be contained Q. Strengthened conjecture by Hayes, Popa (2019): any maximal amenable $Q \subset L\mathbb{F}_n$ is coarse.

• Hayes (2020) reduced PT/coarseness conjectures to proving a certain random matrix limit theorem, a la Voiculescu, Haagerup-Thorbjornsen

The “free complementation” (FC) problem

• Are there maximal amenable vN subalgebras in $L\mathbb{F}_n$ that are not FC ? Are there maximal amenable MASAs that are not FC?

Motivation

• Peterson-Thom conjecture (2011): if $Q \subset M = L\mathbb{F}_n$ is maximal amenable then any $Q_0 \subset L\mathbb{F}_n$ amenable with $Q_0 \cap Q$ diffuse implies must be contained Q. Strengthened conjecture by Hayes, Popa (2019): any maximal amenable $Q \subset L\mathbb{F}_n$ is coarse.

• Hayes (2020) reduced PT/coarseness conjectures to proving a certain random matrix limit theorem, a la Voiculescu, Haagerup-Thorbjornsen

• Belinschi-Capitaine, Bordenave-Collins have recently solved the latter! Thus settling the PT+coarseness conjecture.

The "free complementation" (FC) problem

• Are there maximal amenable vN subalgebras in $L\mathbb{F}_n$ that are not FC ? Are there maximal amenable MASAs that are not FC?

Motivation

• Peterson-Thom conjecture (2011): if $Q \subset M = L\mathbb{F}_n$ is maximal amenable then any $Q_0 \subset L\mathbb{F}_n$ amenable with $Q_0 \cap Q$ diffuse implies must be contained Q. Strengthened conjecture by Hayes, Popa (2019): any maximal amenable $Q \subset L\mathbb{F}_n$ is coarse.

• Hayes (2020) reduced PT/coarseness conjectures to proving a certain random matrix limit theorem, a la Voiculescu, Haagerup-Thorbjornsen

• Belinschi-Capitaine, Bordenave-Collins have recently solved the latter! Thus settling the PT+coarseness conjecture.

• Big problem though: for any freely complemented $Q \subset L\mathbb{F}_n$ one already knows both PT and coarseness hold true, and there are no known examples of maximal amenable $Q \subset L\mathbb{F}_n$ that are not FC!

Candidates for non-FC MASAs in $L\mathbb{F}_n$

Recall that any diffuse amenable $B \subset M = B * N$ is maximal amenable, PT-absorbing and coarse (Popa 1982).

• Let $g \in \mathbb{F}_n$ be so that $g^{\mathbb{Z}}$ is maximal abelian in $\mathbb{F}_n$. Then $A_g = \{u_g\}''$ is maximal amenable in $L\mathbb{F}_n$ (P1982). Is A_g freely complemented in $L\mathbb{F}_n$, even if $g^{\mathbb{Z}}$ is not freely complemented in $\mathbb{F}_n$?

Candidates for non-FC MASAs in $L\mathbb{F}_n$

Recall that any diffuse amenable $B \subset M = B * N$ is maximal amenable, PT-absorbing and coarse (Popa 1982).

- Let $g \in \mathbb{F}_n$ be so that $g^{\mathbb{Z}}$ is maximal abelian in $\mathbb{F}_n$. Then $A_g = \{u_g\}''$ is maximal amenable in $L\mathbb{F}_n$ (P1982). Is A_g freely complemented in $L\mathbb{F}_n$, even if $g^{\mathbb{Z}}$ is not freely complemented in $\mathbb{F}_n$?
- Is the *radial MASA* $L_n \subset L\mathbb{F}_n$, $2 \leq n < \infty$, defined by $L_n = \{\sum_{i=1}^{n}(u_i + u_i^*)\}''$ freely complemented? (NB: L_n is known to be maximal amenable by Cameron-Fang-Ravichandran-White 2010).

Candidates for non-FC MASAs in $L\mathbb{F}_n$

Recall that any diffuse amenable $B \subset M = B * N$ is maximal amenable, PT-absorbing and coarse (Popa 1982).

• Let $g \in \mathbb{F}_n$ be so that $g^{\mathbb{Z}}$ is maximal abelian in $\mathbb{F}_n$. Then $A_g = \{u_g\}''$ is maximal amenable in $L\mathbb{F}_n$ (P1982). Is A_g freely complemented in $L\mathbb{F}_n$, even if $g^{\mathbb{Z}}$ is not freely complemented in $\mathbb{F}_n$?

• Is the *radial MASA* $L_n \subset L\mathbb{F}_n$, $2 \leq n < \infty$, defined by $L_n = \{\sum_{i=1}^n (u_i + u_i^*)\}''$ freely complemented? (NB: L_n is known to be maximal amenable by Cameron-Fang-Ravichandran-White 2010).

• If $\{B_i\}_i$ are diffuse amenable vN in $M = L\mathbb{F}_n$ with B_i freely complemented and $B_i \not\prec_{L\mathbb{F}_n} B_j$, $\forall i \neq j$, then $B = \oplus_i u_i p_i B_i p_i u_i^*$ is maximal amenable in M by (P1982) for any $p_i \in \mathcal{P}(B_i)$ and $u_i \in \mathcal{U}(M)$ satisfying $\sum_i u_i p_i u_i^* = 1$. Is $B \subset M$ freely complemented ?

• Boutonnet-Popa 2022: Let $\{(M_j, \tau_j)\}_{j\in J}$ be tracial vN algebras, with $s_j \in M_j$ semicircular, $\forall j$. Denote ℓ^2_* the set of square summable J-tuples/$\mathbb{R}$ with at least two non-zero entries. For each $t = (t_j)_j \in \ell^2_*$ denote by $A(t)$ the abelian vN generated in $M = *_{j\in J} M_j$ by $s(t) := \sum_j t_j s_j \in M$. Then $A(t)$ is maximal amenable in M, $\forall t \in \ell^2_*$, with $A(t) \prec_M A(t')$ iff $t, t' \in \ell^2_*$ proportional. Are they FC in M?

- Boutonnet-Popa 2022: Let $\{(M_j, \tau_j)\}_{j\in J}$ be tracial vN algebras, with $s_j \in M_j$ semicircular, $\forall j$. Denote ℓ^2_* the set of square summable J-tuples$/\mathbb{R}$ with at least two non-zero entries. For each $t = (t_j)_j \in \ell^2_*$ denote by $A(t)$ the abelian vN generated in $M = *_{j\in J} M_j$ by $s(t) := \sum_j t_j s_j \in M$. Then $A(t)$ is maximal amenable in M, $\forall t \in \ell^2_*$, with $A(t) \prec_M A(t')$ iff $t, t' \in \ell^2_*$ proportional. Are they FC in M?
- Cases of interest for the BP examples: (a) when $M_j = A(s_j) \otimes R$, $\forall j \in J$, where $A(s_j) = \{s_j\}'' \subset M_j$; (b) when $M_j = A(s_j) \rtimes \Gamma_j$, where Γ_j is an amenable group and $\Gamma_j \curvearrowright A_j$ is a trace preserving action, $\forall j \in J$; (c) any situation where M_j is "much bigger" than $A(s_j)$, for instance when M_j abelian non-separable, like $M_j = (L\mathbb{Z})^\omega$, $\forall j$.

- Boutonnet-Popa 2022: Let $\{(M_j, \tau_j)\}_{j\in J}$ be tracial vN algebras, with $s_j \in M_j$ semicircular, $\forall j$. Denote ℓ^2_* the set of square summable J-tuples/$\mathbb{R}$ with at least two non-zero entries. For each $t = (t_j)_j \in \ell^2_*$ denote by $A(t)$ the abelian vN generated in $M = *_{j\in J} M_j$ by $s(t) := \sum_j t_j s_j \in M$. Then $A(t)$ is maximal amenable in M, $\forall t \in \ell^2_*$, with $A(t) \prec_M A(t')$ iff $t, t' \in \ell^2_*$ proportional. Are they FC in M?

- Cases of interest for the BP examples: (a) when $M_j = A(s_j) \otimes R$, $\forall j \in J$, where $A(s_j) = \{s_j\}'' \subset M_j$; (b) when $M_j = A(s_j) \rtimes \Gamma_j$, where Γ_j is an amenable group and $\Gamma_j \curvearrowright A_j$ is a trace preserving action, $\forall j \in J$; (c) any situation where M_j is "much bigger" than $A(s_j)$, for instance when M_j abelian non-separable, like $M_j = (L\mathbb{Z})^\omega$, $\forall j$.

- Noticing that if $M = A^{*n}$ with A non-separable (ns), then $A(t)$ are all separable, it "looks like" in some sense the only ns MASAs in such M are the $A_k = 1 * ... * A * ...1$ (kth position), which would show that rank n is "remembered" by M!

- Boutonnet-Popa 2022: Let $\{(M_j, \tau_j)\}_{j\in J}$ be tracial vN algebras, with $s_j \in M_j$ semicircular, $\forall j$. Denote ℓ^2_* the set of square summable J-tuples$/\mathbb{R}$ with at least two non-zero entries. For each $t = (t_j)_j \in \ell^2_*$ denote by $A(t)$ the abelian vN generated in $M = *_{j\in J} M_j$ by $s(t) := \sum_j t_j s_j \in M$. Then $A(t)$ is maximal amenable in M, $\forall t \in \ell^2_*$, with $A(t) \prec_M A(t')$ iff $t, t' \in \ell^2_*$ proportional. Are they FC in M?

- Cases of interest for the BP examples: (a) when $M_j = A(s_j) \otimes R$, $\forall j \in J$, where $A(s_j) = \{s_j\}'' \subset M_j$; (b) when $M_j = A(s_j) \rtimes \Gamma_j$, where Γ_j is an amenable group and $\Gamma_j \curvearrowright A_j$ is a trace preserving action, $\forall j \in J$; (c) any situation where M_j is "much bigger" than $A(s_j)$, for instance when M_j abelian non-separable, like $M_j = (L\mathbb{Z})^\omega$, $\forall j$.

- Noticing that if $M = A^{*n}$ with A non-separable (ns), then $A(t)$ are all separable, it "looks like" in some sense the only ns MASAs in such M are the $A_k = 1 * ... * A * ...1$ (kth position), which would show that rank n is "remembered" by M!

- I actually tend to believe that any maximal amenable $B \subset L\mathbb{F}_n$ is FC! This would be an amazing structural property of the free group factors.

Non-iso of A^{*n} for non-separable A

Theorem (Boutonnet–Drimbe-Ioana-Popa 03/2023)

Let A be an ns-abelian vN algebra. Then $A^{*n}, 2 \leq n \leq \infty$, are mutually non-isomorphic, with $\mathcal{F}(A^{*n}) = 1$ whenever $n < \infty$.

Non-iso of A^{*n} for non-separable A

Theorem (Boutonnet–Drimbe-Ioana-Popa 03/2023)

Let A be an ns-abelian vN algebra. Then $A^{*n}, 2 \leq n \leq \infty$, are mutually non-isomorphic, with $\mathcal{F}(A^{*n}) = 1$ whenever $n < \infty$.

- We actually show that the ns-free group factors A^{*n} have a special structure that makes them strikingly rigid!

The singular abelian core of a II_1 factor

- Recall that given a tracial vN (M, τ), a weakly closed abelian *-subalgebra $A \subset M$ is *singular* if it has no non-trivial self-intertwiner in M: if $\xi \in 1_A M 1_A$ satisfies $A\xi = \xi A$ then $\xi \in A$. Equivalently, $Ap_1 \not\prec_M Ap_2$, $\forall p_1 \perp p_2 \in \mathcal{P}(A)$. We refer to such subalgebras as *singular abelian*.

The singular abelian core of a II_1 factor

• Recall that given a tracial vN (M, τ), a weakly closed abelian *-subalgebra $A \subset M$ is *singular* if it has no non-trivial self-intertwiner in M: if $\xi \in 1_A M 1_A$ satisfies $A\xi = \xi A$ then $\xi \in A$. Equivalently, $Ap_1 \not\prec_M Ap_2$, $\forall p_1 \perp p_2 \in \mathcal{P}(A)$. We refer to such subalgebras as *singular abelian*.

• We often view a singular abelian $A \subset M$ as its class modulo unitary conjugacy in M.

The singular abelian core of a II_1 factor

• Recall that given a tracial vN (M, τ), a weakly closed abelian *-subalgebra $A \subset M$ is *singular* if it has no non-trivial self-intertwiner in M: if $\xi \in 1_A M 1_A$ satisfies $A\xi = \xi A$ then $\xi \in A$. Equivalently, $Ap_1 \not\prec_M Ap_2$, $\forall p_1 \perp p_2 \in \mathcal{P}(A)$. We refer to such subalgebras as *singular abelian*.

• We often view a singular abelian $A \subset M$ as its class modulo unitary conjugacy in M.

• Two singular abelian $A_1, A_2 \subset M$ are *disjoint* if $A_1 \not\prec_M A_2$. This means there are no non-zero projections $p_i \in \mathcal{P}(A_i)$ such that $A_1 p_1, A_2 p_2$ are unitary conjugate in M (so it is equivalent to $A_2 \not\prec_M A_1$).

The singular abelian core of a II_1 factor

• Recall that given a tracial vN (M, τ), a weakly closed abelian *-subalgebra $A \subset M$ is *singular* if it has no non-trivial self-intertwiner in M: if $\xi \in 1_A M 1_A$ satisfies $A\xi = \xi A$ then $\xi \in A$. Equivalently, $Ap_1 \not\prec_M Ap_2$, $\forall p_1 \perp p_2 \in \mathcal{P}(A)$. We refer to such subalgebras as *singular abelian*.

• We often view a singular abelian $A \subset M$ as its class modulo unitary conjugacy in M.

• Two singular abelian $A_1, A_2 \subset M$ are *disjoint* if $A_1 \not\prec_M A_2$. This means there are no non-zero projections $p_i \in \mathcal{P}(A_i)$ such that $A_1 p_1, A_2 p_2$ are unitary conjugate in M (so it is equivalent to $A_2 \not\prec_M A_1$).

• Let $\mathcal{S}(M)$ be the set of families $\{A_i\}_i$ of mutually disjoint (classes of) singular abelian in a II_1 factor M. For two such families one has $\{A_i^1\}_i \leq \{A_j^2\}_j$ if $\forall i$ and $p \in \mathcal{P}(A_i^1)$, $\exists j$ s.t. $A_i^1 p \prec_M A_j^2$. They are equivalent if one has both $\leq$ and $\geq$ This amounts to: each one can be obtained from the other by unitary conj. + cutting&glueing.

• Taking the II_∞ factor $M \otimes \mathcal{B}(\ell^2 I)$ with I sufficiently large, one can view any $\{A_i\}_i \in \mathcal{S}(M)$ as one single (class of) singular abelian $\oplus_i A_i^0$, where $A_i^0 \sim A_i$. In this "unfolded form", equivalence amounts to unitary conjugacy and $\leq$ amounts to one being equivalent to a reduced of the other. This is the **unfolded form** of $\mathcal{S}(M)$.

• Taking the II_∞ factor $M \otimes \mathcal{B}(\ell^2 I)$ with I sufficiently large, one can view any $\{A_i\}_i \in \mathcal{S}(M)$ as one single (class of) singular abelian $\oplus_i A_i^0$, where $A_i^0 \sim A_i$. In this "unfolded form", equivalence amounts to unitary conjugacy and $\leq$ amounts to one being equivalent to a reduced of the other. This is the **unfolded form** of $\mathcal{S}(M)$.

• $\mathcal{S}(M)/\sim$ clearly inductively ordered with respect to $\leq$ and in fact it has a unique maximal element, which we call the **singular abelian core** of M and denote it $\mathcal{A}_M$. Its unfolded form is thus the (unique up to unitary conjugacy!) singular abelian $\mathcal{A} \subset \mathcal{M} = M \otimes \mathcal{B}(\ell^2 I)$ generated by finite projections with the property that any weakly closed abelian *-subalgebra $B \subset \mathcal{M}$ with $Tr(1_B) < \infty$ and $B \subset 1_B \mathcal{M} 1_B$ singular, satisfies $B \prec_{\mathcal{M}} \mathcal{A}$.

- Taking the II_∞ factor $M \otimes \mathcal{B}(\ell^2 I)$ with I sufficiently large, one can view any $\{A_i\}_i \in \mathcal{S}(M)$ as one single (class of) singular abelian $\oplus_i A_i^0$, where $A_i^0 \sim A_i$. In this "unfolded form", equivalence amounts to unitary conjugacy and $\leq$ amounts to one being equivalent to a reduced of the other. This is the **unfolded form** of $\mathcal{S}(M)$.

- $\mathcal{S}(M)/\sim$ clearly inductively ordered with respect to $\leq$ and in fact it has a unique maximal element, which we call the **singular abelian core** of M and denote it $\mathcal{A}_M$. Its unfolded form is thus the (unique up to unitary conjugacy!) singular abelian $\mathcal{A} \subset \mathcal{M} = M \otimes \mathcal{B}(\ell^2 I)$ generated by finite projections with the property that any weakly closed abelian *-subalgebra $B \subset \mathcal{M}$ with $Tr(1_B) < \infty$ and $B \subset 1_B \mathcal{M} 1_B$ singular, satisfies $B \prec_{\mathcal{M}} \mathcal{A}$.

- The **size** $\mathrm{d}(\mathcal{A}_M)$ of the singular core $\mathcal{A}_M$ is the trace of its support in unfolded form. Alternatively, if $\mathcal{A}_M$ is given in folded form as a family $\{A_i\}_i$ of disjoint singular abelian in M, then $\mathrm{d}(\mathcal{A}_M) = \sum_i \tau(1_{A_i})$.

Note: by P2019 if M separable, then $\mathrm{d}(\mathcal{A}_M) = 2^{\aleph_0}$.

The sans-core and sans-rank of a II_1 factor

- If $A \subset M$ is singular abelian with Ap non-separable $\forall p \in \mathcal{P}(A)$, then we call it a **sans**-subalgebra.

The sans-core and sans-rank of a II_1 factor

- If $A \subset M$ is singular abelian with Ap non-separable $\forall p \in \mathcal{P}(A)$, then we call it a **sans**-subalgebra.

- We obviously have the same considerations for the set $\mathcal{S}_{ns}(M)$ $(\subset \mathcal{S}(M))$ of families of disjoint sans-algebras as for the $\mathcal{S}(M)$. We call the (unique) maximal element in $\mathcal{S}_{ns}(M)/\sim$ the *singular abelian non-separable core* (**sans-core**) of the II_1 factor M and denote it $\mathcal{A}_M^{ns}$. As with the singular abelian core, we view it in either "folded" or "unfolded" form.

The sans-core and sans-rank of a II_1 factor

• If $A \subset M$ is singular abelian with Ap non-separable $\forall p \in \mathcal{P}(A)$, then we call it a **sans**-subalgebra.

• We obviously have the same considerations for the set $\mathcal{S}_{ns}(M)$ $(\subset \mathcal{S}(M))$ of families of disjoint sans-algebras as for the $\mathcal{S}(M)$. We call the (unique) maximal element in $\mathcal{S}_{ns}(M)/\sim$ the *singular abelian non-separable core* (**sans-core**) of the II_1 factor M and denote it $\mathcal{A}_M^{ns}$. As with the singular abelian core, we view it in either "folded" or "unfolded" form.

• The iso-class of the (unique up to unitary conjugacy) sans-core inclusion $\mathcal{A}_M^{ns} \subset \mathcal{M} = M \otimes \mathcal{B}(\ell^2 I)$ is an iso-invariant of M. So its size $\mathrm{d}(\mathcal{A}_M^{ns}) := \sum_{A \in \mathcal{A}_M^{ns}} \tau(1_A)$ is an iso-invariant of M as well.

The sans-core and sans-rank of a II_1 factor

• If $A \subset M$ is singular abelian with Ap non-separable $\forall p \in \mathcal{P}(A)$, then we call it a **sans**-subalgebra.

• We obviously have the same considerations for the set $\mathcal{S}_{ns}(M)$ $(\subset \mathcal{S}(M))$ of families of disjoint sans-algebras as for the $\mathcal{S}(M)$. We call the (unique) maximal element in $\mathcal{S}_{ns}(M)/\sim$ the *singular abelian non-separable core* (**sans-core**) of the II_1 factor M and denote it $\mathcal{A}_M^{ns}$. As with the singular abelian core, we view it in either "folded" or "unfolded" form.

• The iso-class of the (unique up to unitary conjugacy) sans-core inclusion $\mathcal{A}_M^{ns} \subset \mathcal{M} = M \otimes \mathcal{B}(\ell^2 I)$ is an iso-invariant of M. So its size $\mathrm{d}(\mathcal{A}_M^{ns}) := \sum_{A \in \mathcal{A}_M^{ns}} \tau(1_A)$ is an iso-invariant of M as well.

• We call the size $\mathrm{d}(\mathcal{A}_M^{ns})$ of the sans-core the **sans-rank** of the II_1 factor M and denote it $\mathrm{r}_{ns}(M)$.

The sans-core and sans-rank of a II_1 factor

• If $A \subset M$ is singular abelian with Ap non-separable $\forall p \in \mathcal{P}(A)$, then we call it a **sans**-subalgebra.

• We obviously have the same considerations for the set $\mathcal{S}_{ns}(M)$ $(\subset \mathcal{S}(M))$ of families of disjoint sans-algebras as for the $\mathcal{S}(M)$. We call the (unique) maximal element in $\mathcal{S}_{ns}(M)/\sim$ the *singular abelian non-separable core* (**sans-core**) of the II_1 factor M and denote it $\mathcal{A}_M^{ns}$. As with the singular abelian core, we view it in either "folded" or "unfolded" form.

• The iso-class of the (unique up to unitary conjugacy) sans-core inclusion $\mathcal{A}_M^{ns} \subset \mathcal{M} = M \otimes \mathcal{B}(\ell^2 I)$ is an iso-invariant of M. So its size $\mathrm{d}(\mathcal{A}_M^{ns}) := \sum_{A \in \mathcal{A}_M^{ns}} \tau(1_A)$ is an iso-invariant of M as well.

• We call the size $\mathrm{d}(\mathcal{A}_M^{ns})$ of the sans-core the **sans-rank** of the II_1 factor M and denote it $\mathrm{r}_{ns}(M)$.

• One obviously has $\mathrm{r}_{ns}(M^t) = \mathrm{r}_{ns}(M)/t$, $\forall t > 0$!

Sans-core&rank of free product factors

• For an abelian vN algebra A we denote $p_{ns}(A)$ the maximal projection p of A with the property that Ap has no separable summands.

Sans-core&rank of free product factors

- For an abelian vN algebra A we denote $p_{ns}(A)$ the maximal projection p of A with the property that Ap has no separable summands.

Theorem (BDIP 2023)

- Let $A_0, A_1, ...$ be a (possibly finite) sequence of two or more diffuse abelian vN algebras. Then the sans-core of the II_1 factor $M = A_0 * A_1 * ....$ is given by $\{A_k p_{ns}(A_k)\}_{k \geq 0}$ (folded form), with its sans-rank given by $\mathrm{r}_{ns}(M) = \sum_{k \geq 0} \tau(p_{ns}(A_k))$.

Sans-core&rank of free product factors

- For an abelian vN algebra A we denote $p_{ns}(A)$ the maximal projection p of A with the property that Ap has no separable summands.

Theorem (BDIP 2023)

- Let $A_0, A_1, ...$ be a (possibly finite) sequence of two or more diffuse abelian vN algebras. Then the sans-core of the II_1 factor $M = A_0 * A_1 * ....$ is given by $\{A_k p_{ns}(A_k)\}_{k\geq 0}$ (folded form), with its sans-rank given by $\mathrm{r}_{ns}(M) = \sum_{k\geq 0} \tau(p_{ns}(A_k))$.

- If $M = M_1 * M_2 ...$ is a free product of II_1 factors then the sans-core of M is given by $\mathcal{A}_M^{ns} = \cup_k \mathcal{A}_{M_k}^{ns}$ and sans-rank given by $\mathrm{r}_{ns}(M) = \sum_k \mathrm{r}_{ns}(M_k)$.

Sans-core&rank of free product factors

- For an abelian vN algebra A we denote $p_{ns}(A)$ the maximal projection p of A with the property that Ap has no separable summands.

Theorem (BDIP 2023)

- Let $A_0, A_1, ...$ be a (possibly finite) sequence of two or more diffuse abelian vN algebras. Then the sans-core of the II_1 factor $M = A_0 * A_1 * ....$ is given by $\{A_k p_{ns}(A_k)\}_{k\geq 0}$ (folded form), with its sans-rank given by $\text{r}_{ns}(M) = \sum_{k\geq 0} \tau(p_{ns}(A_k))$.

- If $M = M_1 * M_2 ...$ is a free product of II_1 factors then the sans-core of M is given by $\mathcal{A}_M^{ns} = \cup_k \mathcal{A}_{M_k}^{ns}$ and sans-rank given by $\text{r}_{ns}(M) = \sum_k \text{r}_{ns}(M_k)$.

Corollary

- Let $A_0, A_1, A_2, ...$ be diffuse abelian vN algebras, with A_k non-separable if $k \geq 1$, and for each $n \geq 1$ denote $M_n = A_0 * A_1 * .... * A_n$. Then M_n are mutually non-isomorphic II_1 factors, with $\mathcal{F}(M_n) = 1$, $\forall n$.
- In particular, if A is ns-abelian vN then $A^{*n}, 2 \leq n \leq \infty$, are non-isomorphic, with $\mathcal{F}(A^{*n}) = 1$ if $n < \infty$.

About the proof

- *The phenomenology behind these results*: any singular abelian $A \subset M_1 * M_2$ that's "transversal" to both M_1, M_2 must be separable.

About the proof

- *The phenomenology behind these results*: any singular abelian $A \subset M_1 * M_2$ that's "transversal" to both M_1, M_2 must be separable.

More generaly, we have:

Lemma 1

Let M_1, M_2 be diffuse vN algebras and denote $M = M_1 * M_2$. If $B \subset M$ is a weakly closed *-subalgebra such that $B \not\prec_M M_i$, $i = 1, 2$, then $B' \cap 1_B M 1_B$ is separable.

About the proof

- *The phenomenology behind these results*: any singular abelian $A \subset M_1 * M_2$ that's "transversal" to both M_1, M_2 must be separable.

More generaly, we have:

Lemma 1

Let M_1, M_2 be diffuse vN algebras and denote $M = M_1 * M_2$. If $B \subset M$ is a weakly closed *-subalgebra such that $B \not\prec_M M_i$, $i = 1, 2$, then $B' \cap 1_B M 1_B$ is separable.

Lemma 2

With $M = M_1 * M_2$ as above, let $A \subset M$ be singular abelian purely non-separable. Then there exist projections $p_1, p_2 \in A$ such that $p_1 + p_2 = 1_A$ and Ap_i unitary conjugate into M_i, $i = 1, 2$.

What about the "classic" free group factor problem?

• It doesn't seem possible to use the non-iso of A^{*n} for non-separable A to deduce the non-iso of the $L\mathbb{F}_n$. Note however hat if $M = A^{*n}$ with A non-separable, then there exists an increasing net of subfactors $N_i \nearrow M$ such that $N_i \simeq L\mathbb{F}_n$, $\forall i$.

What about the "classic" free group factor problem?

• It doesn't seem possible to use the non-iso of A^{*n} for non-separable A to deduce the non-iso of the $L\mathbb{F}_n$. Note however hat if $M = A^{*n}$ with A non-separable, then there exists an increasing net of subfactors $N_i \nearrow M$ such that $N_i \simeq L\mathbb{F}_n$, $\forall i$.

• By results of Dykema and Radulescu (1992) using Voiculescu's free probability methods, $\text{ng}(L\mathbb{F}_\infty) = \infty$ (so the solution to (3)) implies $L\mathbb{F}_n$, $2 \leq n \leq \infty$, non-isomorphic and $\mathcal{F}(L\mathbb{F}_n) = 1$, $\forall n < \infty$. So (1) and (2) would follow as well.

What about the "classic" free group factor problem?

• It doesn't seem possible to use the non-iso of A^{*n} for non-separable A to deduce the non-iso of the $L\mathbb{F}_n$. Note however hat if $M = A^{*n}$ with A non-separable, then there exists an increasing net of subfactors $N_i \nearrow M$ such that $N_i \simeq L\mathbb{F}_n$, $\forall i$.

• By results of Dykema and Radulescu (1992) using Voiculescu's free probability methods, $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$ (so the solution to (3)) implies $L\mathbb{F}_n$, $2 \leq n \leq \infty$, non-isomorphic and $\mathcal{F}(L\mathbb{F}_n) = 1$, $\forall n < \infty$. So (1) and (2) would follow as well.

• *Tightness conjecture* states that if a II_1 factor M has stably bounded number of generators, $\sup_t \mathrm{ng}(M^t) < \infty$, then M is *tight*, i.e., $\exists R_0, R_1 \subset M$ such that ${}_{R_0}L^2M_{R_1}$ is irreducible. In particular, if M is finitely generated and $\mathcal{F}(M) = 1$, then M would follow tight.
Since $\mathcal{F}(L\mathbb{F}_\infty) \neq 1$ (Voiculescu 1988, Radulescu 1991), this would show that if $\mathrm{ng}(L\mathbb{F}_\infty) < \infty$ then $L\mathbb{F}_\infty$ is tight, contradicting Ge-Popa 1996.
Thus, tightness conjecture implies $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$, more generally $\exists 1 \geq c > 0$ such that $n \geq \mathrm{ng}(L\mathbb{F}_n) \geq cn$, $\forall n$. Thus, by remarks above, the tightness conjecture solves $(1) - (3)$.