The free group factors conundrum

Göteborg, June 19th 2024

to Mikael, on his 65th birthday

Sorin Popa

Background

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- Despite recent progress in classifying large classes of II_1 factors arising from various data (such as groups and their action on spaces), a large number of basic questions concerning the free group factors remained open. Most notably: are the $L(\mathbb{F}_n)$, $2 \leq n \leq \infty$, non-isomorphic?
- This problem gave rise, directly or indirectly, to a huge amount of mathematics, a multitude of concepts and insightful techniques: approximation properties (compact, weak amenability, the Λ-invariant), absorption/repelling/tree-like behavior of $L\mathbb{F}_n$ as bimodule over its amenable subalgebras, free probability methods (random matrix models and free entropy), deformation-rigidity/intertwining methods, W*-boundary methods (bi-exactness and proper proximality), thin&tight decomposition methods, L^2-cohomology attempts, bounded generation, ...

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• The difficulty seems to come from the very nature of $L\mathbb{F}_n$: a subtle mixture between "mild rigidity" (due to spectral gap of $\mathbb{F}_n \curvearrowright L\mathbb{F}_n$ and "tree-ness") and a multitude of deformation properties (free malleability, compact c.p., finite rank c.b.). With both features "spread out" inside $L\mathbb{F}_n$ in a random manner!

Some key open problems about $L\mathbb{F}_n$

Given a II_1 factor M, I'll denote by $\text{ng}(M)$ the minimal number $2 \leq n \leq \infty$ of selfadjoint elements that can generate M as a vN algebra.

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(7) *The derivation/similarity problem for $M = L\mathbb{F}_n$*: do there exist non-inner derivations $\delta : M \to \mathcal{B}(L^2 M \otimes \ell^2\mathbb{N})$?

The “free complementation” (FC) problem

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• Peterson-Thom conjecture (2011): if $Q \subset M = L\mathbb{F}_n$ is maximal amenable then any $Q_0 \subset L\mathbb{F}_n$ amenable with $Q_0 \cap Q$ diffuse must be contained Q. Strengthened conjecture by Hayes, Popa (2019): any maximal amenable $Q \subset L\mathbb{F}_n$ is coarse.

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• One hitch about all this: for any freely complemented $Q \subset L\mathbb{F}_n$ one already knows both PT and coarseness hold true (Popa 82), and there are no known examples of maximal amenable $Q \subset L\mathbb{F}_n$ that are not FC!

Test cases for the FC problem

Recall that for any B diffuse amenable, $B \subset M = B * N$ is maximal amenable, PT-absorbing and coarse.

- Let $g \in \mathbb{F}_n$ be so that $g^{\mathbb{Z}}$ is maximal abelian in $\mathbb{F}_n$. Then $A_g = \{u_g\}''$ is maximal amenable in $L\mathbb{F}_n$ (P1982). Is A_g freely complemented in $L\mathbb{F}_n$, even if $g^{\mathbb{Z}}$ is not freely complemented in $\mathbb{F}_n$?

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- Is the *radial MASA* $L_n \subset L\mathbb{F}_n$, $2 \leq n < \infty$, defined by $L_n = \{\sum_{i=1}^n (u_i + u_i^*)\}''$ freely complemented? (NB: L_n is known to be maximal amenable by Cameron-Fang-Ravichandran-White 2010).

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• If $\{B_i\}_i$ are diffuse amenable vN in $M = L\mathbb{F}_n$ with B_i freely complemented and $B_i \not\prec_{L\mathbb{F}_n} B_j$, $\forall i \neq j$, then $B = \oplus_i u_i p_i B_i p_i u_i^*$ is maximal amenable in M for any $p_i \in \mathcal{P}(B_i)$ and $u_i \in \mathcal{U}(M)$ satisfying $\sum_i u_i p_i u_i^* = 1$. Is $B \subset M$ freely complemented ? Check this for $L\mathbb{F}_n = B_1 * ... * B_n$.

This latter “re-patching” test case has in fact been recently settled:

Theorem (Boschert-Davis-Hiatt 06/2024)

Let $M = A_1 * \cdots * A_n$ with $n \geq 2$ and A_i diffuse abelian tracial vN $\forall i$. Let $p_i \in \mathcal{P}(A_i)$ and $\{u_i\}_i \subset \mathcal{U}(M)$ be such that $\sum_j u_j p_j u_j^* = 1$. Then $B := \sum_j u_j A_j p_j u_j^*$ is freely complemented in M.

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• I think it is possible that the FC problem has a positive answer, i.e., that any maximal amenable $B \subset L\mathbb{F}_n$ is FC. This would of course be a rather amazing structural phenomenon about the free group factors! The following weaker form should definitely hold true:

The weak FC conjecture

Given any amenable $B \subset L\mathbb{F}_n$ there exists a Haar unitary $u \in L\mathbb{F}_n$ that's free independent to B.

A new test case: the B-P examples

• Boutonnet-Popa 2022: Let $\{(M_j, \tau_j)\}_{j\in J}$ be tracial vN algebras, with $s_j \in M_j$ semicircular, $\forall j$. Denote ℓ^2_* the set of square summable J-tuples$/\mathbb{R}$ with at least two non-zero entries. For each $t = (t_j)_j \in \ell^2_*$ denote by $A(t)$ the abelian vN generated in $M = *_{j\in J} M_j$ by $s(t) := \sum_j t_j s_j \in M$. Then $A(t)$ is maximal amenable in M, $\forall t \in \ell^2_*$, with $A(t) \prec_M A(t')$ iff $t, t' \in \ell^2_*$ proportional. Is it FC in M if all M_j amen?

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- Cases of interest for the B-P examples: (a) when $M_j = A(s_j) \otimes R$, $\forall j \in J$, where $A(s_j) := \{s_j\}'' \subset M_j$; (b) when $M_j = A(s_j) \rtimes \Gamma_j$, where Γ_j is an amenable group and $\Gamma_j \curvearrowright A_j$ is a trace preserving action, $\forall j \in J$; (c) any situation where M_j is "much bigger" than $A(s_j)$, such as when $M_j = A_j$ abelian purely non-separable $\forall j$.

Non-iso of A^{*n} for non-separable A

• The B-P examples and the proofs involved naturally lead to the problem of whether any purely non-separable (singular) MASA B in $M = A^{*n}$, with A purely non-separable is "made up" of pieces of $A_k := 1 * ... * A * ...1$ (kth position), a fact that would imply that n is "remembered" by the iso-class of A^{*n}! Indeed one has:

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Theorem (Boutonnet–Drimbe-Ioana-Popa 03/2023)

Let A be a non-separable tracial vN algebra. Then $A^{*n}, 2 \leq n \leq \infty$, are mutually non-isomorphic, with $\mathcal{F}(A^{*n}) = 1$ whenever $n < \infty$.

About the proof

• To prove this result one considers the *singular abelian non-separable core* (**sans-core**) $\mathcal{A}_M^{ns}$ of a II_1 factor M, as the maximal singular abelian purely non-separable wo-closed *-subalgebra generated by finite projections in the II_∞ factor $\mathcal{M} = M \otimes \mathcal{B}(\ell^2 I)$, where $I = 2^{|M_h|}$. One easily sees that $\mathcal{A}_M^{ns}$ is unique in $\mathcal{M}$ up to unitary conjugacy. So the trace Tr of the support of $\mathcal{A}_M^{ns}$ in $\mathcal{M}$ is an iso-invariant of M as well, which we call the **sans-rank** of the II_1 factor M and denote it $\text{r}_{ns}(M)$.

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• One then proves that any wo-closed singular abelian $B \subset M := M_0 * M_1$ that's "transversal" to both M_1, M_2 must be separable. And thus any purely non-separable $A \subset M$ must "split" as $A = Ap_0 + Ap_1$, where $p_0, p_1 \in \mathcal{P}(A)$, $p_0 + p_1 = 1_A$, and Ap_i unitary conjugate into M_i, $i = 0, 1$.

What about the “classic” free group factor problem?

• It doesn't seem possible to use the non-iso of A^{*n} for non-separable A to deduce the non-iso of the $L\mathbb{F}_n$. But note hat if $M = A^{*n}$ with A purely ns, then $\exists N_i \nearrow M$ subfactors such that $N_i \simeq L\mathbb{F}_n$, $\forall i$.

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• By results of Dykema, Radulescu (1992) using Voiculescu's free probability, if $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$ (so if (3) holds true) then $L\mathbb{F}_n$, $2 \leq n \leq \infty$, non-iso and $\mathcal{F}(L\mathbb{F}_n) = 1$, $\forall n < \infty$. So (1) and (2) would follow as well.

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• *Tightness conjecture* states that if a II_1 factor M has stably bounded number of generators, i.e. $\sup_t \mathrm{ng}(M^t) < \infty$, then M is *R-tight*: $\exists R_0, R_1 \subset M$ such that ${}_{R_0}L^2M_{R_1}$ is irreducible. In particular, if M is finitely generated and $\mathcal{F}(M) \neq 1$, then M would follow R-tight. Since $\mathcal{F}(L\mathbb{F}_\infty) \neq 1$ (Voiculescu 1988, Radulescu 1991), this would show that if $\mathrm{ng}(L\mathbb{F}_\infty) < \infty$ then $L\mathbb{F}_\infty$ is tight, contradicting Ge-Popa 1996.

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- Thus, tightness conjecture implies $\mathrm{ng}(L\mathbb{F}_\infty) = \infty$, more generally $\exists 1 \geq c > 0$ such that $n \geq \mathrm{ng}(L\mathbb{F}_n) \geq cn$, $\forall 2 \leq n \leq \infty$. So by the remarks above, the tightness conjecture solves (1), (2), (3).