Some open problems in W*-rigidity

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Questions on Cartan decomposition

• Find classes of factors $L^\infty(X) \rtimes \Gamma$ with unique Cartan subalgebras (up to unitary conjugacy), or merely unique group measure space Cartan decomposition (...).

• Does $L^\infty(\mathbb{T}^n) \rtimes SL(n, \mathbb{Z})$ have unique Cartan decomposition, $\forall n \geq 3$? Note that if so, then the action $SL(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$ would follow W*-Super-rigid (by Furman 99).

Conjecture : If Γ is an arbitrary non-amenable group and $\Gamma \curvearrowright X$ Bernoulli, then $L^\infty(X) \rtimes \Gamma$ has unique Cartan, up to unitary conj.

• Are there free mixing p.m.p. group actions $\Gamma \curvearrowright X$ with Γ non-amenable, such that $L^\infty(X) \rtimes \Gamma$ doesn't have unique Cartan decomposition ?...

• Construct factors with exactly n unitary conjugacy classes of Cartan subalgebras, for some given $n \geq 2$.

Questions on Cartan-rigidity for groups

• Characterize the class of $\mathcal{C}$-rigid groups, i.e. groups Γ with the property that $L^\infty(X) \rtimes \Gamma$ has unique Cartan subalgebra (up to unitary conjugacy) $\forall \Gamma \curvearrowright X$ free ergodic.

• Does $L(\Gamma)$ strongly solid imply Γ is $\mathcal{C}$-rigid ?

Conjecture (Popa-Vaes): If $\beta_1^{(2}(\Gamma) \neq 0$ (more generally, if $\beta_n^{(2}(\Gamma) \neq 0$, for some $n \geq 1$), then Γ is $\mathcal{C}$-rigid. Note that if so, then $\beta_n^{(2)}(\Gamma)$ would follow an isomorphism invariant for $L^\infty(X) \rtimes \Gamma$

Questions on cocycle and OE-superrigidity

• Find classes of OE superrigid & cocycle superrigid (CSR) group actions (with targets in $\mathcal{U}_{fin}$, $\mathcal{U}_{dis}$, etc).

• What are the groups Γ for which $\exists\ \Gamma \curvearrowright X$ CSR ($\mathcal{U}_{fin}$, $\mathcal{U}_{dis}$, etc)?

• Find the class $\mathcal{CS}$ of groups Γ such that any Bernoulli Γ-action is $\mathcal{U}_{fin}$-CSR, or $\mathcal{U}_{dis}$-CSR. The conjecture is that $\Gamma \in \mathcal{CS}$ iff $\beta_1^{(2)}(\Gamma) = 0$ (Peterson-Sinclair: $\beta_1^{(2)}(\Gamma) \neq 0$ implies Bernoulli $\Gamma \curvearrowright X$ are not $\mathbb{T}$-CSR; also partial results for the converse)

• Find larger classes $\mathcal{U}$ of "target" groups with the property that any Bernoulli action of a Kazhdan (or other) group is $\mathcal{U}$-CSR.

Higher cohomology groups of eq. rel.

- Calculate $\mathrm{H}^2(\mathcal{R}_\Gamma)$ more generally $\mathrm{H}^n(\mathcal{R}_\Gamma)$ for some $\Gamma \curvearrowright X$, e.g. for Bernoulli. No such calculations exist for $n \geq 2$! For Γ Kazhdan and action Bernoulli, one expects $\mathrm{H}^n(\mathcal{R}_\Gamma) = \mathrm{H}^n(\Gamma)$.

Questions on the fundamental group (Popa-Vaes)

For Γ countable group, denote $\mathcal{S}_{factor}(\Gamma) = \{\mathcal{F} \subset \mathbb{R}_+ \mid \exists \Gamma \curvearrowright X$ free erg with $\mathcal{F}(L^\infty(X) \rtimes \Gamma) = \mathcal{F}\}$. Similarly $\mathcal{S}_{eqrel}(\Gamma)$.

• Axiomatize subgroups $\mathcal{F} \subset \mathbb{R}_+$ for which $\exists$ separable II_1 factor M, (resp eq rel $\mathcal{R}$) such that $\mathcal{F}(M) = \mathcal{F}$ (resp $\mathcal{F}(\mathcal{R}) = \mathcal{F}$). Polishable+Borel?

• Calculate $\mathcal{S}_{eqrel}(\mathbb{F}_\infty), \mathcal{S}_{factor}(\mathbb{F}_\infty)$

• $\mathcal{S}_{factor}(\Gamma) \subset \mathcal{S}_{factor}(\mathbb{F}_\infty) = \mathcal{S}_{eqrel}(\mathbb{F}_\infty)$, $\forall \Gamma$? $\mathcal{F}(M) \in \mathcal{S}_{factor}(\mathbb{F}_\infty)$, $\forall M$ separable II_1?

• $\mathcal{S}_{factor}(\Gamma) \subset \mathcal{P}(\mathbb{Q}_+)$, $\forall \Gamma$ ICC with (T)?

• $\{1\} \in \mathcal{S}_{factor}(\Gamma)$, $\forall \Gamma$ non-amenable ?

• Is $\mathcal{F}(L^\infty(X) \rtimes \Gamma) = 1$, for any Bernoulli action of a fin gen (or merely $\beta_1^{(2)}(\Gamma) < \infty$) non-amenable group Γ ? Is it true that if $\Gamma \curvearrowright X$ is Bernoulli, then $\mathcal{F}(L(L^\infty(X) \rtimes \Gamma))$ is either $\{1\}$ or $\mathbb{R}_+$, $\forall \Gamma$?

Free Group Factor Problems

TheNon − isomorphismProblem : $L(\mathbb{F}_n) \simeq L(\mathbb{F}_m) \Rightarrow n = m$?

• More generally, recalling that by Radulescu, Dykema we have $L(\mathbb{F}_n)^t \simeq L(\mathbb{F}_m)^s$ whenever $(n-1)/t^2 = (m-1)/s^2$ and defining $L(\mathbb{F}_x) := L(\mathbb{F}_n)^t$, where $x = (n-1)/t^2 + 1$, is it true that $L(\mathbb{F}_x) \simeq L(\mathbb{F}_y)$ implies $x = y$? Note that by Radulescu, Dykema, if $L(\mathbb{F}_x) \not\simeq L(\mathbb{F}_y)$ for some $1 < x < y \leq \infty$, then all $L(\mathbb{F}_x)$, $1 < x \leq \infty$ are non-isomorphic.

FiniteGenerationProblem : Can $L(\mathbb{F}_\infty)$ be fin gen as a vN algebra ? Do there exist $L(\Gamma)$ which cannot be fin gen ?

Abstract characterizations of $L(\mathbb{F}_n)$

• (Peterson-Thom) Is it true that whenever $B_i \subset L(\mathbb{F}_n)$ amenable with $\cap_i B_i$ diffuse, implies $\vee_i B_i$ amenable ?

• Is it true that any subfactor $M \subset L(\mathbb{F}_n)$ is either amenable or isomorhic to some $L(\mathbb{F}_t)$, $1 < t \leq \infty$?

• Assume a non-amenable II_1 factor M has the property that the "free flip" $x * y \mapsto y * x$ is path connected to id in $\mathrm{Aut}(M * M)$ (or even stronger, that M is *free malleable*). Does this imply $M \simeq L(\mathbb{F}_t)$, some $1 < t \leq \infty$?

Related questions If M factor and classic flip on $M \otimes M$ is path connected to id (or even malleable), then $M \simeq R$? Also: is R free malleable?...

Connes' Rigidity (CR) conjecture

Classic form : If Γ, Λ ICC groups with property (T), does $L(\Gamma) \simeq L(\Lambda)$ imply $\Gamma \simeq \Lambda$?

Strong form : If Γ ICC with prop (T) and Λ ICC, then any $\theta : L(\Gamma) \simeq L(\Lambda)^t$ forces $t = 1$ and $\exists \delta : \Gamma \to \Lambda$, $\gamma \in \mathrm{Hom}(\Gamma, \mathbb{T})$ such that $\theta(\Sigma_g c_g u_g) = \Sigma_g \gamma(g) c_g u_{\delta(g)}$?

Special cases : Check that $L(\Gamma_n) \simeq L(\Gamma_m) \implies n = m$, for $\Gamma_n = PSL(n, \mathbb{Z})$, or for $\Gamma_n = \mathbb{Z}^n \rtimes SL(n, \mathbb{Z})$. (True for $\Gamma_n \subset Sp(n, 1)$ by Cowling-Haagerup).

- Is $L(SL(3, \mathbb{Z}))$ solid ? (Note that $L(PSL(n, \mathbb{Z}))$ are not solid for $n \geq 4$)

- Prove CR conj "up to finite classes", i.e. that $\Gamma \mapsto L(\Gamma)$ if finite to 1.

- Given Γ ICC with (T), $\exists n$ s.t. if $L(\Gamma) = N_1 \otimes ... \otimes N_k$ then $k \leq n$?

- If Γ ICC with property (T), then $\mathcal{F}(L(\Gamma)) = 1$? (Note: this is what strong form of CR conj implies!)

Connes' Approximate Embedding (CAE) conjecture

CAE conjecture: If M is a separable II_1 factor (more generally, a separable finite vN algebra), then $M \hookrightarrow R^\omega$. Equivalently, $M \hookrightarrow \Pi_\omega M_{n\times n}(\mathbb{C})$.

• Does the CAE conj hold for $M = L(\Gamma)$, $\forall\Gamma$ countable group? (N.B.: This is equivalent to the fact that any Γ can be faithfully represented into $\Pi_\omega M_{n\times n}(\mathbb{C})$).

• Is it true that any countable Γ is *sofic*, i.e., it can be represented into the normalizer of $\Pi_\omega D_n$ in $\Pi_\omega M_{n\times n}(\mathbb{C})$, so that to act freely on $\Pi_\omega D_n$, where $D_n \subset M_{n\times n}(\mathbb{C})$ is the diagonal Cartan subalgebra (equivalently, into the normalizer of D^ω in R^ω, so that to act freely on D^ω, where $D \subset R$ is the Cartan subalgebra)