

LECTURES ON ERGODIC THEORY OF GROUP ACTIONS
(A VON NEUMANN ALGEBRA APPROACH)

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1. GROUP ACTIONS: BASIC PROPERTIES

1.1. Probability spaces as von Neumann algebras. The “classical” measure theoretical approach to the study of actions of groups on the probability space is equivalent to a “non-classical” operator algebra approach due to a well known observation of von Neumann, showing that measure preserving isomorphisms between standard probability spaces (X, μ) are in natural correspondence with $*$ -algebra isomorphisms between their function algebras $L^\infty X = L^\infty(X, \mu)$ preserving the functional given by the integral, $\tau_\mu = \int \cdot d\mu$. More precisely:

1.1.1. Theorem. *1°. Let $T : (X, \mu) \rightarrow (Y, \nu)$ be a measurable map with $\nu \circ T = \mu$. Then $\rho_T : L^\infty Y \rightarrow L^\infty X$ defined by $\rho_T(x)(s) = x(Ts), s \in X$, is an injective $*$ -algebra morphism satisfying $\tau_\mu \circ \rho_T = \tau_\nu$. Conversely, if $(X, \mu), (Y, \nu)$ are probability spaces and $\rho : L^\infty Y \rightarrow L^\infty X$ is an injective $*$ -algebra morphism such that $\tau_\mu \circ \rho = \tau_\nu$, then there exists a measurable map $T : X \rightarrow Y$, such that $\rho = \rho_T$. Moreover, T is unique and onto, modulo a set of measure 0, and the correspondence $T \mapsto \rho_T$ is “contravariant” functorial, i.e. $\rho_{S \circ T} = \rho_T \circ \rho_S$. Also, T is a.e. 1 to 1 if and only if ρ is onto and if this is the case then T^{-1} is also measurable and measure preserving.*

2°. If (X, μ) is a non-atomic probability space then $(X, \mu) \simeq (\mathbb{T}, \lambda)$ and $(L^\infty X, \tau_\mu) \simeq (L^\infty \mathbb{T}, \tau_\lambda)$.

Proof. The fact that ρ_T is a $*$ -algebra isomorphism preserving the integral is trivial by the definition. Also, $T \mapsto \rho_T$ is clearly functorial.

If (X, μ) has no atoms then one can easily construct recursively finite “diadic” partitions $P_n = \{p_k^n \mid 1 \leq k \leq m_n\}$ with projections in $L^\infty X$ such that $\tau_\mu(p_k^n) = 2^{-m_n}, \forall k, P_n \subset P_{n+1}, \forall n$, and $\overline{\cup_n \sum_k \mathbb{C}p_k^n} = L^\infty X$, thus giving an isomorphism ρ of $(L^\infty X, \tau_\mu)$

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onto $(L^\infty Y, \tau_\lambda)$, where Y is the compact group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ with its Haar measure λ . Also, this isomorphism clearly implements a measurable onto (a.e.) map $T : X \rightarrow Y$ such that $\rho = \rho_T$. From this point on, the only non-trivial part in completing the proof of both 1° and 2° is then to show that T follows 1 to 1 a.e. as well (by using the fact that ρ_T is onto). We refer to Royden's "Real Analysis" book for the proof of this latter fact. Q.E.D.

From now on, if $T : (X, \mu) \simeq (Y, \nu)$ is an isomorphism of probability spaces then we denote ϑ_T the integral preserving isomorphism of $L^\infty X$ onto $L^\infty Y$ given by $\vartheta_T = \rho_{T^{-1}}$. Note that the correspondence $T \rightarrow \vartheta_T$ becomes "covariant" functorial.

There are two norms on $L^\infty X$ that are relevant for us, namely the ess-sup norm $\|\cdot\| = \|\cdot\|_\infty$ and the norm $\|\cdot\|_2$. Note that the unit ball $(L^\infty X)_1$ of $L^\infty X$ (in the norm $\|\cdot\|$) is complete in the norm $\|\cdot\|_2$. At times, we will also consider the norm $\|\cdot\|_1$ on $L^\infty X$. By Cauchy-Schwartz, we have $\|x\|_1 \leq \|x\|_2 \leq \|x\|_1^{1/2}$, $\forall x \in (L^\infty X)_1$, so the corresponding topologies structures are equivalent.

We often identify $L^\infty X$ with the von Neumann algebra of (left) multiplication operators $L_x, x \in L^\infty X$, where $L_x(\xi) = x\xi, \xi \in L^2 X$. The identification $x \mapsto L_x$ is a *-algebra morphism, it is isometric (from $L^\infty X$ with the ess-sup norm into $\mathcal{B}(L^2 X)$ with the operatorial norm) and takes the $\|\cdot\|_2$ -topology of $(L^\infty X)_1$ onto the strong operator topology on the image. Also, the integral $\tau_\mu(x)$ becomes the vector state $\langle L_x(1), 1 \rangle$, $x \in L^\infty X$. Moreover, if $T : (X, \mu) \simeq (Y, \nu)$ for some other probability space (Y, ν) , then ϑ_T extends to an (isometric) isomorphism of Hilbert spaces $U_T : L^2 X \simeq L^2 Y$ which conjugates the von Neumann algebras $L^\infty X \subset \mathcal{B}(L^2 X)$, $L^\infty Y \subset \mathcal{B}(L^2 Y)$ onto each other, spatially implementing the isomorphism ϑ_T , i.e. $U_T L_x U_T^* = L_{\vartheta_T(x)}, \forall x \in L^\infty X$.

If $\{(X_n, \mu_n)\}_n$ is a sequence of standard probability spaces then the product probability space $\Pi_n(X_n, \mu_n)$ can be defined in the obvious way. One can readily see that there is a natural identification between $L^\infty \Pi_n X_n$ with the integral given by the product measure and the tensor product of algebras $\overline{\otimes}_n(L^\infty X_n, \tau_{\mu_n})$.

1.2. Actions of groups by automorphisms. We denote by $\text{Aut}(X, \mu)$ the group of (classes modulo null sets of) *measure preserving automorphisms* $T : (X, \mu) \simeq (X, \mu)$ of the standard probability space (X, μ) . Denote $\text{Aut}(L^\infty X, \tau_\mu)$ the group of *-automorphisms of the von Neumann algebra $L^\infty X$ that preserve the functional $\tau_\mu = \int \cdot d\mu$, and identify $\text{Aut}(X, \mu)$ and $\text{Aut}(L^\infty X, \tau_\mu)$ via the map $T \mapsto \vartheta_T$ described in 1.1.1° above.

One immediate benefit of the functional analysis framework and of this identification is that it gives a natural Polish group topology on $\text{Aut}(X, \mu)$, given by pointwise $\|\cdot\|_2$ -convergence in $\text{Aut}(L^\infty X, \tau_\mu)$, i.e. $\vartheta_n \rightarrow \vartheta$ in $\text{Aut}(L^\infty X, \tau_\mu)$ if $\lim_n \|\vartheta_n(x) - \vartheta(x)\|_2 = 0, \forall x \in L^\infty X$.

1.2.1. Lemma. 1°. *The topologies of wo, so and so* convergence on the unitary group $\mathcal{U}(\mathcal{H})$ on a Hilbert space \mathcal{H} coincide and give a structure of topological group on*

$\mathcal{U}(\mathcal{H})$. If \mathcal{H} is separable then $\mathcal{U}(\mathcal{H})$ endowed with either one of these topologies is a Polish group.

2°. The map $\text{Aut}(X, \mu) \ni T \mapsto U_T \in \mathcal{U}(L^2 X)$ has close image and it is an isomorphism of topological groups, from $\text{Aut}(X, \mu)$ onto its image in $\mathcal{U}(L^2 X)$.

Proof. If $u_i, u \in \mathcal{U}(\mathcal{H})$ are unitary elements such that $\lim_i \|u_i \xi - u \xi\| = 0, \forall \xi \in \mathcal{H}$, then taking $\xi = u^* \eta$ we get

$$\lim_i \|u_i \xi - u \xi\| = \lim_i \|u_i u^* \eta - \eta\| = 0, \forall \eta \in \mathcal{H}.$$

Thus, the *so* and *so** topologies coincide on $\mathcal{U}(\mathcal{H})$. Also, if u_i tends to u in the *wo* topology then for any unit vector $\xi \in \mathcal{H}$ we have $\langle u_i \xi, u \xi \rangle \rightarrow 1$, thus $\|u_i \xi - u \xi\|^2 = 2 - 2\text{Re}\langle u_i \xi, u \xi \rangle \rightarrow 0$, showing that u_i converges *so* to u . The rest of the statement is trivial by the definitions. Q.E.D.

An *action* of a discrete group Γ on the standard probability space (X, μ) is a group morphism $\sigma : \Gamma \rightarrow \text{Aut}(X, \mu)$. We'll often use the notation $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ to emphasize an action σ , or simply $\Gamma \curvearrowright X$ if no confusion is possible. We'll sometimes consider topological groups G other than discrete (typically locally compact or Polish), in which case an action of G on (X, μ) will be a morphism of topological groups $G \rightarrow \text{Aut}(X, \mu)$.

Using the identification between $\text{Aut}(X, \mu)$ and $\text{Aut}(L^\infty X, \tau_\mu)$, we alternatively view σ as an action of Γ on $(L^\infty X, \tau_\mu)$, i.e as a group morphism $\sigma : \Gamma \rightarrow \text{Aut}(L^\infty X, \tau_\mu)$. Although we use the same notation for both actions, the difference will be clear from the context. Furthermore, when viewing σ as an action on the probability space (X, μ) , we'll use the simplified notation $\sigma_g(t) = gt$, for $g \in \Gamma, t \in X$. The relation between σ as an action on (X, μ) and respectively on $(L^\infty X, \tau_\mu)$ is then given by the equations $\sigma_g(x)(t) = x(g^{-1}t), \forall t \in X$ (a.e.), which hold true for each $g \in \Gamma, x \in L^\infty X$.

Since any σ_g extends to a unitary operator on $L^2 X$, $\sigma : \Gamma \rightarrow \text{Aut}(L^\infty X, \tau_\mu)$ extends to a unitary representation of Γ on the Hilbert space $L^2 X$, denoted U_σ , or simply σ .

Two actions $\sigma : \Gamma \rightarrow \text{Aut}(X, \mu), \theta : \Lambda \rightarrow \text{Aut}(Y, \nu)$, are *conjugate with respect to* an isomorphism $\delta : \Gamma \simeq \Lambda$ if there exists an isomorphism of probability spaces $\Delta : (X, \mu) \rightarrow (Y, \nu)$ (or equivalently an integral preserving isomorphism Δ from $L^\infty X$ onto $L^\infty Y$) such that $\Delta \circ \sigma_g = \theta_{\delta(g)} \circ \Delta, \forall g \in \Gamma$. If there exists some $\delta : \Gamma \simeq \Lambda$ such that $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ are conjugate with respect to δ , then we simply say that $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ are *conjugate*. Note that if σ, θ are faithful then this condition is equivalent to the condition $\{\Delta \sigma_g \Delta^{-1} \mid g \in \Gamma\} = \{\theta_h \mid h \in \Lambda\}$.

1.3. Freeness, ergodicity and mixing properties. The action $\Gamma \curvearrowright X$ is *free* if for any $g \in \Gamma, g \neq e$, the set $\{t \in X \mid gt = t\}$ has μ -measure 0. On the function space $L^\infty X$, this amounts to $a\sigma_g(x) = xa, \forall x \in L^\infty X$, for some $a \in L^\infty X$, implies either $g = e$ or $a = 0$.

The action is *ergodic* if $X_0 \subset X$ measurable with $gX_0 = X_0$ (a.e.) for all $g \in \Gamma$, implies $X_0 = X$ or $X_0 = \emptyset$ (a.e.), in other words $\mu(X_0) = 0, 1$. Equivalently, if p is a projection in the von Neumann algebra $L^\infty X$ then $\sigma_g(p) = p, \forall g \in \Gamma$ implies $p = 0, 1$. It is immediate to see that if this condition is satisfied then the only measurable functions $x : X \rightarrow \mathbb{C}$ that are fixed by σ , i.e. $\sigma_g(x) = x, \forall g \in \Gamma$, are the constant functions (a.e.). Indeed, this is clear for $x \in L^\infty X$, because if σ fixes x then it fixes all its spectral decomposition (this being obtained as weak limits of polynomials in x, x^* , which are all fixed by σ), which thus follow scalar multiples of 1, thus x itself is constant. If x is arbitrary measurable then σ fixes its polar decomposition $x = ua$, i.e. $\sigma_g(u) = u, \sigma_g(a) = a, \forall g \in \Gamma$, where $a = (x^*x)^{1/2}$ and $u = xa^{-1}$. Thus $u \in \mathbb{C}$ by the first part, and $b = (1 + a)^{-1} \in L^\infty X_+$ satisfies $\sigma_g(b) = b, \forall g \in \Gamma$, implying $b \in \mathbb{C}$, thus $a \in \mathbb{C}$. Note that the above argument also shows that the fixed point algebra $(L^\infty X)^\sigma$ is $\|\cdot\|_2$ -dense (resp. $\|\cdot\|_1$ -dense) in the set of fixed points of the action σ on $L^2 X$ (resp. $L^1 X$).

Notice that an action $\Gamma \curvearrowright^\sigma (X, \mu)$ is ergodic iff the corresponding unitary representation σ of Γ on $L^2 X \ominus \mathbb{C}1$ has no fixed vectors, i.e. it does not contain the trivial representation 1_Γ .

1.3.1. Lemma. 1° . *A unitary representation $\sigma : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ is ergodic (i.e. $1_\Gamma \notin \sigma$) iff given any $\xi, \eta \in \mathcal{H}$ and any $\varepsilon > 0$ there exists $g \in \Gamma$ such that $\operatorname{Re}\langle \sigma_g(\xi), \eta \rangle \leq \varepsilon$.*

2° . *An action $\Gamma \curvearrowright^\sigma (X, \mu)$ is ergodic iff for any $p, q \in \mathcal{P}(L^\infty X)$ and any $\varepsilon > 0$ there exists $g \in \Gamma$ such that $\tau_\mu(\sigma_g(p)q) \leq (1 + \varepsilon)\tau_\mu(p)\tau_\mu(q)$.*

Proof. 1° . If $\operatorname{Re}\langle \sigma_g(\xi), \eta \rangle \geq \varepsilon, \forall g \in \Gamma$, then $\operatorname{Re}\langle \zeta, \eta \rangle \geq \varepsilon, \forall \zeta \in K_\varepsilon = \operatorname{co}^w\{\sigma_g(\xi) \mid g \in \Gamma\}$. Since K_ε is convex and compact in the w^* duality topology, it follows that there exists a unique element $\zeta_0 \in K_\varepsilon$ of minimal norm. But K_ε is σ -invariant by definition and $\|\sigma_g(\zeta)\| = \|\zeta\|, \forall \zeta \in K_\varepsilon$, so by uniqueness $\sigma_g(\zeta_0) = \zeta_0, \forall g \in \Gamma$. Since σ is ergodic, this implies $\zeta_0 = 0 \in K_\varepsilon$. Thus $0 = \operatorname{Re}\langle \zeta_0, \eta \rangle \geq \varepsilon$, a contradiction.

2° . Just apply part 1° to $\xi = p - \tau_\mu(p)1, \eta = q - \tau_\mu(q)1$.

Q.E.D.

1.3.2. Remarks. (a). The proof of part 1° of the above lemma works equally well when instead of a unitary representation σ one has a semigroup $T_h, h \in H$, of contractions on the Hilbert space \mathcal{H} , i.e. $T_h T_{h'} = T_{hh'}, h, h' \in H$, and $\|T_h\| \leq 1, \forall h$.

(b). Assume σ is as in 1° of the lemma (or more generally an ergodic semigroup of contractions on \mathcal{H}). Since the closure of convex subsets of \mathcal{H} in the w^* -duality topology and in the Hilbert norm coincide and since the proof of 1.3.1.1 $^\circ$ above shows that $0 \in K_\varepsilon$, it follows that for any $\varepsilon > 0$ there exist $g_1, \dots, g_n \in \Gamma$ such that

$$(b') \quad \|n^{-1} \sum_i \sigma_{g_i}(\xi)\| \leq \varepsilon.$$

On the other hand, this latter fact trivially implies 1.3.1.1 $^\circ$. More concrete ways of getting convex combinations of $\sigma_{g_i}(\xi)$ of small norm are provided by the following general form of von Neumann's Mean Ergodic theorem: If $\{T_h \mid h \in H\}$ is an amenable

semigroup of contractions on the Hilbert space \mathcal{H} , and $F_i \subset \Gamma$ is a (right) Folner family of finite subsets of the amenable semigroup H , then

$$(b'') \quad \lim_i \left\| |F_i|^{-1} \sum_{g \in F_i} T_g(\xi) - p_0(\xi) \right\| = 0, \forall \xi \in \mathcal{H},$$

where p_0 is the orthogonal projection onto the subspace of fixed points $\mathcal{H}_0 = \{\xi \in H \mid T_g \xi = \xi, \forall g \in H\}$. The proof of this result goes as follows: Since T_g are contractions, any fixed point of T_g is a fixed point for T_g^* , thus $\mathcal{H} \ominus \mathcal{H}_0$ is invariant to $T_g, g \in H$. So to prove the theorem it is sufficient to prove it in the case $\mathcal{H}_0 = 0$. But then the span of the vectors $\xi = \eta - T_h(\eta), \eta \in \mathcal{H}$, is dense in \mathcal{H} , showing that it is sufficient to prove the convergence (b'') for such $\eta - T_h(\eta)$ only. But this is trivial by the Følner condition.

If the group Γ in 1.3.1.1° above is finitely generated, say by $g_1, \dots, g_k \in \Gamma$, and we denote by T the Laplacian $k^{-1} \sum_h \sigma_{g_i} \in \mathcal{B}(\mathcal{H})$, then by von Neumann's ergodic mean value theorem (for the semigroup $H = \{n \mid n \geq 1\}$) it follows that for each $\xi \in \mathcal{H}, \varepsilon > 0$ there exists n large enough such that $\|T^n \xi\| \leq \varepsilon$. In case Γ is itself amenable with Folner sets $F_n \subset \Gamma$, one can take the "specific" convex combinations in (b') of the form $|F_n|^{-1} \sum_{g \in F_n} \sigma_g(\xi)$.

(c). The above considerations applied to $q = 1 - p$ in part 1° of Lemma 1.3.1 show that if the group Γ is infinite and σ is an arbitrary m.p. action of Γ on the probability space (X, μ) then given any subset of positive measure $X_0 \subset X$ there exist an infinite sequence $g_n \in \Gamma$ such that $\limsup_n \mu(X_0 \cap g_n X_0) \geq \mu(X_0)^2$, i.e. a version for arbitrary groups of Poincaré's "returning" lemma. We leave this as an exercise. There is a straight way to get recursively a "returning" sequence g_n when the group Γ satisfies the following property: For any finite $F_0 \subset \Gamma$ there exists a semigroup $\Gamma_0 \subset \Gamma$ disjoint from F_0 that generates Γ as a group. For if we assume Γ satisfies this property, then for any $\xi \in L^2 X$ the set $K_\xi^{\Gamma_0} = \{\sigma_g(\xi) \mid g \in \Gamma_0\}$ is Γ_0 -invariant thus its (unique) element ξ_0 of minimal norm $\|\cdot\|_2$ is a Γ_0 -fixed point. Since Γ_0 generates Γ , ξ_0 is fixed by Γ as well. Applying this to $\xi_0 = 1 - p$ and reasoning as in the proof of 1.3.1, for a given $\varepsilon > 0$ one gets $g \in \Gamma_0$ (thus not in F) such that $\tau_\mu(\sigma_g(1-p)p) \leq (1+\varepsilon)\tau_\mu(E(1-p)p)$, where E is the conditional expectation onto the fixed points of σ . But $\tau_\mu(E(1-p)p) \leq \tau_\mu(1-p)\tau_\mu(p)$ (exercise) so $\tau_\mu(\sigma_g(p)p) \geq (1+\varepsilon)\tau(p)^2 - \varepsilon\tau(p)$, which for $\varepsilon \leq \tau(p)^2/4$ is larger than $(1 - \varepsilon^{1/2})\tau(p)^2$.

An action $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ is *weak mixing* (resp. *mixing*) if for any finite set $F \subset L^\infty X \ominus \mathbb{C}$ and any $\varepsilon > 0$ there exists $g \in \Gamma$ (resp. there exists $K_0 \subset \Gamma$ finite) such that $|\tau_\mu(\eta^* \sigma_g(\xi))| \leq \varepsilon, \forall \xi, \eta \in F$ (resp. $\forall g \in K_0$). It is trivial to see that if this condition holds true for subsets F in $L^\infty X \ominus \mathbb{C}1$ then it holds true for subsets F in $L^2 X \ominus \mathbb{C}1$ as well.

Related to this, we'll say that a unitary representation of Γ on a Hilbert space \mathcal{H} is *weak mixing* (resp. *mixing*) if $\forall F \subset \mathcal{H}$ finite there exists $g \in \Gamma$ (resp. there exists

$K_0 \subset \Gamma$ finite) such that $|\langle \sigma_g(\xi), \eta \rangle| \leq \varepsilon, \forall \xi, \eta \in F$ (resp. $\forall g \in K_0$). We will use the same terminology for orthogonal representations of Γ on real Hilbert spaces (i.e. group morphisms of Γ into the group of orthogonal operators on a real Hilbert space \mathcal{H} ; note that such representations correspond, via GNS construction, to positive definite real valued maps on Γ).

It is immediate to see that a representation σ is mixing iff all its coefficients vanish at infinity, equivalently all positive definite functions affiliated with the representation are in $c_0(\Gamma)$ (are compact). For weak-mixing, we have some alternative characterisations:

1.3.3. Lemma. *Let $\sigma : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ on the Hilbert space \mathcal{H} . The following are equivalent:*

- (i). σ is weak mixing.
- (ii). Given any other representation $\sigma_0 : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_0)$ the product representation $\sigma \otimes \sigma_0$ of the group Γ is ergodic.
- (iii). $\sigma(\Gamma)' \cap \mathcal{K}(\mathcal{H}) = 0$.
- (iii'). There are no $\sigma(\Gamma)$ -invariant finite dimensional non-zero vector subspaces of \mathcal{H} .

Proof. (i) \implies (ii). Assume $\xi \in \mathcal{H} \otimes \mathcal{H}_0$ is fixed by $\sigma \otimes \rho$. By the density of $\mathcal{H} \otimes \mathcal{H}_0$ in $\mathcal{H} \otimes \mathcal{H}_0$ there exists an orthonormal system $\xi_1, \xi_2, \dots, \xi_n \in \mathcal{H}$ and elements $\eta_1, \eta_2, \dots, \eta_n \in \mathcal{H}_0$ such that if we denote $\xi' = \sum_{i=1}^n \xi_i \otimes \eta_i$ then we have

$$\|\xi - \xi'\|_2 < \varepsilon / (3\|\xi\|_2) \quad \text{and} \quad \|\xi'\|_2 \leq \|\xi\|_2.$$

Since σ is weakly mixing, there exists $g \in \Gamma$ such that

$$\sum_{i,j=1}^n |\langle (\sigma_g(\xi_i)\xi_j^*) | \langle (\rho_g(\eta_i)\eta_j^*) \rangle| < \varepsilon/3.$$

Thus, in $\mathcal{H} \otimes \mathcal{H}_0$ we have:

$$|\langle (\sigma_g \otimes \rho_g)(\xi'^*) \xi' \rangle| < \varepsilon/3.$$

As a consequence we get

$$\|\xi\|_2^2 \leq |\langle (\sigma_g \otimes \rho_g)(\xi'^*) \xi' \rangle| + 2\|\xi - \xi'\|_2 \|\xi\|_2 < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\xi = 0$.

(ii) \implies (iii). If $\sigma(\Gamma)$ commutes with a non-zero compact operator then it commutes with a non-zero finite rank projection. By applying (ii) to $\rho = \sigma$ and by taking into account that the Hilbert space of Hilbert-Schmidt operators S on \mathcal{H} , with the action $S \mapsto \text{Ad}_{\sigma_g}(S)$ of Γ on it, can be naturally identified with $\mathcal{H} \otimes \mathcal{H}^*$, with the action $\sigma \otimes \sigma$ on it, it follows that there are no finite rank projections commuting with $\sigma(\Gamma)$.

(iii) \Leftrightarrow (iii'). If $\mathcal{H}_0 \subset \mathcal{H}$ is finite-dimensional invariant then the projection on it is compact and commutes with $\sigma(\Gamma)$. Conversely if $K \in \sigma(\Gamma)' \cap \mathcal{K}(\mathcal{H})$ then any spectral projection of K^*K still commutes with $\sigma(\Gamma)$, so the subspace it projects on is $\sigma(\Gamma)$ -invariant.

(iii) \implies (i). Let $\mathcal{H}_0 \subset \mathcal{H}$ be the linear span of F and p_0 the orthogonal projection onto \mathcal{H}_0 , regarded as an element in the Hilbert space \mathcal{HS} of Hilbert-Schmidt operators on \mathcal{H} . Since by (ii), \mathcal{H} has no non-zero finite dimensional subspaces invariant to σ it follows that $\forall \delta > 0, \exists g \in \Gamma$ such that in \mathcal{HS} we have $Tr(\sigma_g(p_0)p_0) < \delta$. Indeed, because if there would exist some $\delta_0 > 0$ such that $Tr(\sigma_g(p_0)p_0) \geq \delta_0, \forall g \in \Gamma$, then for any y in the weak closure of the convex hull $K_{p_0} \subset \mathcal{HS}$ of $\{\sigma_g(p_0)\}_g$ we would still have $Tr(y p_0) \geq \delta_0$.

In particular, this would happen for the unique element $y_0 \in K_{p_0}$ of minimal norm $\| \cdot \|_{2,Tr}$. But since $\| \sigma_g(y_0) \|_{2,Tr} = \| y_0 \|_{2,Tr}$, it follows that $\sigma_g(y_0) = y_0, \forall g \in \Gamma$. This implies that any spectral projection of $y_0 \geq 0$ is invariant to σ . By (ii) any such projection is equal to 0. Thus $y_0 = 0$, contradicting $Tr(y_0 p_0) \geq \delta_0 > 0$. But if $Tr(\sigma_g(p_0)p_0) < \delta$ for some $g \in G$ and for a sufficiently small $\delta > 0$, then this implies $\langle \sigma_g(\xi)\eta \rangle < \varepsilon, \forall \xi, \eta \in F = F^*$. Q.E.D.

1.3.4. Corollary. *If Γ is an infinite group then the left regular representation λ_Γ of Γ is mixing and all infinite dimensional irreducible representations of Γ are weakly mixing.*

Note that (i) \Leftrightarrow (iii') in Lemma 1.3.3 holds true for real orthogonal representations of Γ as well, as its proof doesn't depend on \mathcal{H} being complex or real Hilbert space. Likewise, 1.3.4 equally holds for orthogonal representations of Γ .

Lemma 1.3.3 also implies:

1.3.5. Corollary. *Let $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ be an action of a discrete group Γ on the standard probability space (X, μ) . The following conditions are equivalent:*

- (i). σ is weakly mixing.
- (ii). For any action $\Gamma \overset{\rho}{\curvearrowright} (Y, \nu)$, the fixed point algebra of the product action $\sigma_g \otimes \rho_g, g \in \Gamma$, coincides with the fixed point algebra of ρ , i.e. $L^\infty(X \times Y, \mu \times \nu)^{\theta \otimes \rho} = \mathbb{C} \otimes L^\infty(Y, \nu)^\rho$.
- (iii). For any ergodic action $\Gamma \overset{\rho}{\curvearrowright} (Y, \nu)$, the product action $\sigma_g \otimes \rho_g, g \in \Gamma$, is ergodic.
- (iv). The only finite dimensional vector subspace of $L^2 X$ invariant to $\sigma_g, g \in G$, is $\mathbb{C}1$.

1.4. Compact actions. An action $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ is compact if the closure of $\sigma(\Gamma)$ in $\text{Aut}(X, \mu)$ is compact. Similarly, a representation $\sigma : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ is compact if $\sigma(\Gamma)$ is precompact in $\mathcal{U}(\mathcal{H})$ (the later with its Polish group topology).

1.4.1. Lemma. *The following conditions are equivalent.*

(i). σ is compact.

(ii). $\sigma(\Gamma)\xi$ is precompact in $\mathcal{H}, \forall \xi \in \mathcal{H}$.

(iii). The von Neumann algebra generated by $\sigma(\Gamma)$ is atomic of type I_{fin} , i.e. σ is a direct sum of finite dimensional representations.

Proof. This is well known (and trivial).

Q.E.D.

Note that a compact representation (or action) can be ergodic, or even strongly ergodic (see examples ...), but not weakly mixing. Even more so, if ρ is compact and σ weak mixing then ρ cannot be contained (direct summand) in σ , i.e. a compact rep cannot be contained in a w-mixing rep.

1.5. Strong ergodicity and spectral gap. An action $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ has spectral gap if there exist $g_1, g_2, \dots, g_n \in \Gamma$ and $c > 0$ such that $\sum_i \|\sigma_{g_i}(\xi) - \xi\|_2 \geq c\|\xi\|_2, \forall \xi \in L^2 X \setminus \mathbb{C}1$. The convexity properties of the Hilbert space $L^2 X$ easily imply that this is equivalent to the existence of a gap $(c_0, 1)$ in the spectrum of the (norm one selfadjoint) Laplace operator $\xi \mapsto (2n)^{-1} \sum_i (\sigma_{g_i}(\xi) + \sigma_{g_i}^{-1}(\xi))$ on $L^2 X \ominus \mathbb{C}1$, with $\mathbb{C}1$ as the eigenspace for the eigenvalue 1.

A bounded sequence $(x_n)_n \subset L^\infty X$ is asymptotically σ -invariant if $\lim_n \|\sigma_g(x_n) - x_n\|_2 = 0, \forall g \in \Gamma$. The asymptotically invariant sequence (x_n) is non-trivial if $\liminf_n \|x_n - \tau(x_n)1\|_2 \neq 0$. The action $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ is strongly ergodic if it has no non-trivial asymptotically invariant sequences. Spectral gap clearly implies strong ergodicity. We have the following equivalent characterisations of each of these properties.

1.5.1. Proposition. *The following are equivalent:*

(i). The action $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ has spectral gap.

(ii). There exist $c_0 > 0$ and $g_1, \dots, g_n \in \Gamma$ such that for any projection $p \in L^\infty X$ with $\tau_\mu(p) \leq 1/2$ we have $\sum_i \|\sigma_{g_i}(p) - p\|_2 \geq c_0 \|p\|_2$.

(ii'). For any $1 > \delta_0 > 0$ there exists $c > 0$ and $g_1, \dots, g_n \in \Gamma$ such that for any projection $p \in L^\infty X$ with $\tau_\mu(p) \leq \delta_0$ we have $\sum_i \|\sigma_{g_i}(p) - p\|_2 \geq c \|p\|_2$.

(iii). There exists a countable subgroup $\Gamma_0 \subset \Gamma$ for which there is no $\sigma(\Gamma_0)$ -invariant state on $L^\infty X$ other than $\tau_\mu = \int \cdot d\mu$.

(iv). $1_\Gamma \notin (\Gamma \curvearrowright L^2 X \ominus \mathbb{C}1)$.

Proof. (i) \Leftrightarrow (iv) and (ii') \Rightarrow (ii) are trivial.

(iii) \Rightarrow (ii'). This amounts to showing that if we assume (ii') is not satisfied then given any countable subgroup $\Gamma_0 \subset \Gamma$ there exists a $\sigma(\Gamma_0)$ -invariant state on $L^\infty X$ other than τ_μ .

But non-(ii') implies that there exists $1 > \delta_0 > 0$ such that given any countable $\Gamma_0 \subset \Gamma$ there exists a sequence of non-zero projections $p_n \in L^\infty X$ such that $\tau_\mu(p_n) \leq \delta_0$ and $\lim_n \|\sigma_g(p_n) - p_n\|_2 / \|p_n\|_2 = 0, \forall g \in \Gamma_0$. We call such a sequence a $\sigma(\Gamma_0)$ -invariant sequence of projections. By taking a subsequence of p_n if necessary, we may assume

$\tau_\mu(p_n)$ is convergent. We want to prove that we may further assume $\lim_n \tau_\mu(p_n) = 0$. Note that if $c \geq 0$ is the infimum over all $s \geq 0$ for which there exists a $\sigma(\Gamma_0)$ -invariant sequence p_n with $\lim_n \tau_\mu(p_n) = s$ then there exists a $\sigma(\Gamma_0)$ -invariant sequence q_n with $\lim_n \sigma_\mu(q_n) = c$. If $c > 0$ then by ergodicity there exist finite sets $F_n \subset \Gamma_0$ such that $\| |F_n|^{-1} \sum_{g \in F_n} \sigma_g(q_n) - \tau(q_n)1 \|_2 \leq 2^{-n}$. By the asymptotic invariance of q_n , there exists a fast growing $k_1 \ll k_2 \ll \dots$ such that $\| |F_n|^{-1} \sum_{g \in F_n} \sigma_g(q_{k_n}) - q_{k_n} \|_2 \leq 2^{-n}$. But then an easy calculation shows that $p_n = q_n q_{k_n}$ is still $\sigma(\Gamma_0)$ -invariant and $\lim_n \tau_\mu(p_n) = c^2$. Thus $c = 0$. Note that this also proves (ii) \Rightarrow (ii').

By taking a subsequence of the $\sigma(\Gamma_0)$ -invariant sequence p_n we may even assume $\tau_\mu(p_n) \leq 2^{-n}$. But then it is immediate to see that any weak limit point of $\tau_\mu(\cdot p_n) / \tau_\mu(p_n) \in L^1 X \subset (L^\infty X)^*$ is a singular $\sigma(\Gamma_0)$ -invariant state on $L^\infty X$ (because its support has arbitrarily small size).

(iv) \Rightarrow (iii) Assume by contradiction that for any countable subgroup $\Gamma_0 \subset \Gamma$ there exists a $\sigma(\Gamma_0)$ -invariant state $\varphi \neq \tau_\mu$ on $L^\infty X$. We will show that this implies that for any $g_1, \dots, g_n \in \Gamma$ and any $\varepsilon > 0$ there exists a unit vector $\xi \in L^2 X \ominus \mathbb{C}1$ such that $\| \sigma_{g_i}(\xi) - \xi \|_2 \leq \varepsilon, \forall i$.

We first show that we may assume φ is singular (with respect to μ). If φ would be normal then by the Radon-Nykodim theorem it is of the form $\varphi = \tau_\mu(\cdot a)$ for some $a \in L^1 X_+$ with $\tau_\mu(a) = 1, a \neq 1$. But then $a^{1/2} - \tau_\mu(a^{1/2})1 \neq 0$ is a $\sigma(\Gamma_0)$ -invariant vector in $L^1 X \ominus \mathbb{C}1$ and since Γ_0 was arbitrary this would imply that 1_Γ is contained in the representation σ of Γ on $L^2 X \ominus \mathbb{C}1$, a contradiction. Thus, φ is not normal and so its singular part is non-zero and still $\sigma(\Gamma_0)$ -invariant.

Let $f \in L^\infty X$ be a non-zero projection such that $\varphi(f) = 1$ and $\tau_\mu(f) \leq \varepsilon$. Denote \mathcal{L} the set of normal states ψ on $L^\infty X$ such that $\psi(f) \geq 1 - \varepsilon$. Thus \mathcal{L} is a subset of the unit ball of $L^1 X_+$, and note right away that φ is in the closure of \mathcal{L} in $(L^\infty X)^*$ in the duality topology with $L^\infty X$. Let \mathcal{V} denote the set of n -tuples $(\psi - \psi \circ \sigma_{g_i})_{i=1}^n$, with $\psi \in \mathcal{L}$. Then \mathcal{V} is a (bounded) convex subset of $(L^1 X)^n \subset (L^\infty X^*)^n = (L^n \text{fty} X^n)^*$. We claim that $0 = (0, \dots, 0)$ is in the (norm) closure of \mathcal{V} in the Banach space $(L^1 X)^n$. Indeed, for if not then by the Hahn-Banach theorem there exists $\psi = (x_1, \dots, x_n) \in ((L^1 X)^n)^* = (L^1 X^*)^n = (L^\infty X)^n$ such that

$$\operatorname{Re} \sum_{i=1}^n (\psi(x_i) - \psi(\sigma_{g_i}(x_i))) \geq \alpha > 0, \forall \psi \in \mathcal{L}.$$

But this would then hold true equally well for all weak limits of $\psi \in \mathcal{L}$ in $L^\infty X^*$, thus for φ . But φ is σ_{g_i} -invariant, thus

$$0 = \operatorname{Re} \sum_{i=1}^n (\varphi(x_i) - \varphi(\sigma_{g_i}(x_i))) \geq \alpha > 0,$$

a contradiction.

Thus, since $0 \in \overline{\mathcal{L}}$ it follows that there is $a \in L^1 X_+$ with $\tau_\mu(a) = 1, \tau_\mu(a f) \geq 1 - \varepsilon$ and $\| \sigma_{g_i}(a) - a \|_1 \leq \varepsilon^2$. But then $b = (f a)^{1/2} - \tau_\mu((f a)^{1/2})1 \in L^2 X \ominus \mathbb{C}1$ satisfies

$$\| b \|_2 = \tau(f a) - \tau_\mu(f a^{1/2})^2 \geq 1 - \varepsilon - \tau(f) \tau(a) \geq 1 - 2\varepsilon,$$

(by Cauchy-Schwartz), while

$$\|\sigma_{g_i}(b) - b\|_2 \leq \|\sigma_{g_i}(fa) - fa\|_1^{1/2} \leq (\|\sigma_{g_i}(a) - a\|_1 + 2\varepsilon)^{1/2} \leq 2\varepsilon^{1/2}$$

Altogether, $b_0 = b/\|b\|_2$ is a unit vector in $L^2X \ominus \mathbb{C}1$ with $\|\sigma_{g_i}(b_0) - b_0\|_2 \leq 2\varepsilon^{1/2}/(1 - 2\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we are done.

(ii) \Rightarrow (iv). If (iv) doesn't hold true and $\xi \in L^2X \ominus \mathbb{C}1$ is a unit vector ε -invariant to σ_{g_i} for some finite set $g_1, \dots, g_n \in \Gamma$ then either the real or the imaginary part of ξ will still be σ_{g_i} -invariant and be of L^2 -norm $\geq 1/2$. Thus, we may assume $\xi = \xi^*$. This implies $|\xi|$ is almost invariant as well, thus so are ξ_+, ξ_- , while $\tau_\mu(\xi_+) = \tau_\mu(\xi_-)$ (because $\tau_\mu(\xi) = 0$) and $\tau_\mu(\xi_+^2) + \tau_\mu(\xi_-^2) = 1$. We may assume ξ_+ has support s of smaller size than the support of ξ_- . Then if $\tau_\mu(\xi_+^2)$ is "sizeable", an appropriate spectral projection of ξ_+ will be almost invariant and of trace $\leq 1/2$, by Lemma 1.5.2 below. If in turn $\tau_\mu(\xi_+^2)$ is small then $\tau(\xi_+)$ is small, so $\tau(|\xi|) = 2\tau(\xi_+)$ is altogether small. Thus an appropriate spectral projection of $|\xi|$ will be away from 0 and still be almost invariant by Lemma 1.5.2 again. Q.E.D.

1.5.2. Lemma (Namioka's trick). *Let (Y, ν) be a measurable space and $a, b_i \in L^1Y$, $1 \leq i \leq n$, with $a, b_i \geq 0$, $\tau_\nu(a) = 1$, $\Sigma_i \|a - b_i\|_1 < \varepsilon$. Then there exists $s > 0$ such that $\Sigma_i \|e_s(a) - e_s(b_i)\|_2^2 < \varepsilon \|e_s(a)\|_2^2$, where for a measurable function $b : Y \rightarrow [0, \infty)$ and $s > 0$ we denote $e_s(b)$ the characteristic function of the set $\{t \in Y \mid b(t) > s\}$.*

Proof. Note first that if $t, t_i \geq 0$ then $|t - t_i| = \int_{s>0} |\chi_{[s, \infty)}(t) - \chi_{[s, \infty)}(t_i)| ds$. Applying this to $t = a(x), t_i = b_i(x)$, by Fubini's theorem we have

$$\begin{aligned} \Sigma_i \|a - b_i\|_1 &= \Sigma_i \int_Y |a(x) - b_i(x)| d\nu(x) \\ &= \Sigma_i \int_Y \left(\int_{s>0} |e_s(a(x)) - e_s(b_i(x))| ds \right) d\nu(x) \\ &= \Sigma_i \int_{s>0} \left(\int_Y |e_s(a(x)) - e_s(b_i(x))| d\nu(x) \right) ds \\ &= \Sigma_i \int_{s>0} \|e_s(a) - e_s(b_i)\|_1 ds. \end{aligned}$$

In particular, this also shows that $1 = \|a\|_1 = \int_{s>0} \|e_s(a)\|_1 ds$. Thus we have

$$\Sigma_i \int_{s>0} \|e_s(a) - e_s(b_i)\|_1 ds < \varepsilon \int_{s>0} \|e_s(a)\|_1 ds,$$

which implies that for at least one $s > 0$ we have $\Sigma_i \|e_s(a) - e_s(b_i)\|_1 < \varepsilon \|e_s(a)\|_1$. Since for any partial isometry v (which both $e_s(a)$ and $e_s(a) - e_s(b_i)$ are) we have $\|v\|_1 = \|v\|_2^2$, the statement follows. Q.E.D.

1.5.3. Proposition. *The following are equivalent:*

- (i). *The action $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ is strongly ergodic.*
- (ii). *There exist $g_1, \dots, g_n \in \Gamma$ and $1/2 \geq c_0 > 0$ such that for any projection $p \in L^\infty X$ with $c_0 \leq \tau_\mu(p) \leq 1 - c_0$ we have $\sum_i \|\sigma_{g_i}(p) - p\|_2 \geq c_0 \|p\|_2$.*
- (iii). *If ω is a free ultrafilter on \mathbb{N} and $A^\omega = \ell^\infty(\mathbb{N}, L^\infty X) / \mathcal{I}_\omega$, where $\mathcal{I}_\omega = \{(x_n)_n \in \ell^\infty(\mathbb{N}, L^\infty X) \mid \lim_\omega \tau_\mu(x_n^* x_n) = 0\}$, then the action implemented by σ on A^ω by $\sigma_g((x_n)_n) = (\sigma_g(x_n))_n$ is ergodic.*

Moreover, if either of these conditions doesn't hold, then the fixed point algebra of A^ω under the Γ -action is diffuse (has no minimal projections). Equivalently, given any $0 \leq c \leq 1$ there exists an asymptotically invariant sequence p_n with $\lim_n \tau(p_n) = c$.

Proof. The proofs are the same as the proofs in 1.4.1.

Q.E.D.

1.5.4. Remark. We should mention that the only way in which group actions have been shown strongly ergodic in various concrete examples (so far) was by proving they have spectral gap. It is an open problem whether there exist strongly ergodic actions that have no spectral gap. This problem is very similar to a problem of Effros on whether there exist non- Γ group von Neumann II_1 factors $L\Lambda$ (in the sense of Murray and von Neumann) from groups Λ that are inner amenable.

2. EXAMPLES

2.1. Group-like actions. Let α be an automorphism of the discrete abelian group H . Then α implements an automorphism σ^α on the dual \hat{H} of H , preserving the Haar measure λ , by $\sigma^\alpha(\chi) = \chi \circ \alpha^{-1}$, $\chi \in \hat{H}$. Thus, if we identify H with its bidual, i.e. with the dual of \hat{H} and view elements h in H as functions u_h on \hat{H} , thus as elements in $L^\infty(\hat{H})$, then as an automorphism on $L^\infty(\hat{H})$, σ^α acts by $\sigma^\alpha(u_h) = u_{\alpha(h)}$, $h \in H$. Note that, any $h \in H, h \neq e$, satisfies $\tau_\lambda(u_h) = \int h d\lambda = 0$ and that in fact $\{u_h\}_{h \in \Gamma}$ gives an orthonormal basis for $L^2(\hat{H})$.

2.1.1. Lemma. *If $|\{\alpha(h)h^{-1} \mid h \in H\}| = \infty$ then σ^α is properly outer.*

Proof. If σ^α acts as the identity on a set of positive measure $X_0 \subset \hat{H}$ then there exists $h_0 \in H$ such that $c = \tau_\lambda(\chi_{X_0} u_{h_0}) \neq 0$. Since $\sigma^\alpha(u_h)\chi_{X_0} = u_h\chi_{X_0}, \forall h$, this implies $c = \tau_\lambda(u_h\chi_{X_0}u_{h_0}u_{h^{-1}}) = \tau_\lambda(\chi_{X_0}(\sigma^\alpha(u_h)u_{h_0}u_{h^{-1}}))$, i.e. χ_{X_0} has infinitely many Fourier coefficients equal to $c \neq 0$, contradicting the fact that $\chi_{X_0} \in L^\infty(\hat{H}) \subset L^2(\hat{H})$. Q.E.D.

Let now $\alpha : \Gamma \rightarrow \text{Aut}(H)$ be a group morphism and denote by σ^α the action it implements on (\hat{H}, λ) by $\sigma_g^\alpha = \sigma^{\alpha(g)}, g \in \Gamma$.

2.1.2. Lemma. *The following conditions are equivalent:*

- (i). σ^α is ergodic.
- (ii). σ^α is weakly mixing.
- (iii). α has no finite invariant subsets $\neq \{e\}$.
- (iv). For any finite subset $S \subset H$ there exists $h \in \Gamma$ such that $\alpha_h(S) \cap S = \emptyset$.
- (v). The orbit of every element $h \in H \setminus \{e\}$ is infinite.

Proof. (ii) \implies (i) and (iv) \Leftrightarrow (v) are trivial.

(i) \implies (iii). If $\alpha_h(S) = S, \forall h \in \Gamma_0$ for some finite set $S \subset G$ with $e \notin S$, then $x = \sum_{h \in S} u_h \notin \mathbb{C}1$ satisfies $\sigma_\alpha(g)(x) = x, \forall g \in \Gamma$, implying that σ_α is not ergodic.

(iii) \implies (iv). If $\alpha_h(S) \cap S \neq \emptyset, \forall h \in \Gamma$, for some finite set $S \subset G \setminus \{e\}$, then denote by f the characteristic function of S regarded as an element of $\ell^2(H)$. If we denote by $\tilde{\alpha}$ the action (=representation) of Γ on $\ell^2(H)$ implemented by α , then we have $\langle \tilde{\alpha}_g(f), f \rangle \geq 1/|S|, \forall g \in \Gamma$. Thus, the element a of minimal norm $\| \cdot \|_2$ in the weak closure of $\text{co}\{\tilde{\alpha}_g(f) \mid g \in \Gamma\} \subset \ell^2(H)$ is non zero. But then any “level set” of $a \geq 0$ is invariant to α , showing that (c) doesn’t hold true.

(iv) \implies (ii). Let E_0 be a finite set in the unit ball of $L^\infty(\hat{H}) = \{u_h\}''$, $\varepsilon > 0$ and $F_0 \subset \Gamma \setminus \{e\}$ a finite set as well. Let $S_0 \subset H \setminus \{e\}$ be finite and such that $\|(x - \tau(x)1) - x_{S_0}\|_2 \leq \varepsilon/2, \forall x \in E_0$, where x_{S_0} is the orthogonal projection of $\ell^2(H)$ onto $\ell^2(S_0)$. By applying the hypothesis to $S = \cup\{\alpha_g(S_0) \mid g \in F_0\}$, it follows that there exists $g \in \Gamma_0$ such that $\alpha_g(S) \cap S = \emptyset$. But then $g \notin F_0$ and $\alpha_g(S_0) \cap S_0 = \emptyset$. Also, by Cauchy-Schwartz, for each $x, y \in E_0$ we have:

$$|\tau(\sigma^\alpha(g)(x)y) - \tau(x)\tau(y)|$$

$$\begin{aligned} &\leq \|(x - \tau(x)1) - x_{S_0}\|_2 \|y\|_2 + \|(y - \tau(y)1) - y_{S_0}\|_2 \|x\|_2 + |\tau(\sigma^\alpha(g)(x_{S_0})y_{S_0})| \\ &= \|(x - \tau(x)1) - x_{S_0}\|_2 \|y\|_2 + \|(y - \tau(y)1) - y_{S_0}\|_2 \|x\|_2 \leq \varepsilon. \end{aligned}$$

Q.E.D.

2.1.3. Lemma. *Let Γ be a non-amenable group and $\{H_i\}_i$ the family of amenable subgroups of Γ . Then the trivial representation of Γ is not weakly contained in $\oplus_i \ell^2(\Gamma/H_i)$ (thus not weakly contained in $\oplus_i \ell^2(\Gamma/H_i) \overline{\otimes} \ell^2(\mathbb{N})$ either).*

Proof. This follows immediately from the continuity of induction of representations. Indeed, every $\ell^2(\Gamma/H_i)$ is equivalent to the induced from H_i to Γ of the trivial representation 1_{H_i} of H_i , $\text{Ind}_{H_i}^\Gamma 1_{H_i}$. Since H_i is amenable, 1_{H_i} follows weakly contained in the left regular representation λ_{H_i} of H_i . Thus, $\text{Ind}_{H_i}^\Gamma 1_{H_i}$ is weakly contained in $\text{Ind}_{H_i}^\Gamma(\lambda_{H_i})$, which in turn is just the left regular representation λ_Γ of Γ . Altogether, this shows that if 1_Γ is weakly contained in $\oplus_i \ell^2(\Gamma/H_i)$ then it is weakly contained in a multiple of λ_Γ . Since the latter is weakly equivalent to λ_Γ , 1_Γ follows weakly contained in λ_Γ , implying that Γ is amenable, a contradiction. Q.E.D.

Let now Γ act by automorphisms on a discrete abelian group H and denote by σ the action it implements on (\hat{H}, λ) , and thus on $L^\infty(\hat{H}, \lambda)$, then note that the ensuing representation of Γ on $L^2(\hat{H}) \ominus \mathbb{C}1 = \ell^2(H \setminus \{e\})$ is equal to $\oplus_h \ell^2(\Gamma/\Gamma_h)$, where $\Gamma_h \subset \Gamma$ denotes the stabilizer of $h \in H \setminus \{e\}$, $\Gamma_h = \{\gamma \in \Gamma \mid \gamma(h) = h\}$. Lemma 2.1.3 thus shows:

2.1.4. Corollary. *Assume Γ is non-amenable and the stabilizer Γ_h of each $h \in H \setminus \{e\}$ is amenable. For any non-amenable $\Gamma_0 \subset \Gamma$ the action $\sigma|_{\Gamma_0}$ of Γ on (\hat{H}, λ) has a spectral gap, and thus is strongly ergodic. In particular we have:*

1°. *If $\Gamma_0 \subset SL(2, \mathbb{Z})$ is non-amenable then the restriction to Γ_0 of the canonical action of $SL(2, \mathbb{Z})$ on (\mathbb{T}^2, λ) is strongly ergodic.*

2°. *If Γ is an arbitrary non-amenable group and H_0 is a non-trivial, countable discrete abelian group and H denotes the direct sum of infinitely many copies of H_0 indexed by Γ , then the action of Γ by Bernoulli shifts on $\hat{H}_0^\Gamma = \hat{H}$ has spectral gap.*

Finally, note that if one takes $H = \Gamma$ and let Γ act on itself by conjugation, then Lemma 2.1.3 implies that if Γ is non-amenable and the commutant in Γ of any $h \in \Gamma \setminus \{e\}$ is amenable then Γ is not inner amenable either.

2.2. Bernoulli actions. Let (X_0, μ_0) be a standard probability space. Let Γ be a countable discrete group and K a countable set on which Γ acts (by permutations of the set K). Let $(X, \mu) = \prod_k (X_0, \mu_0)_k$ be the standard probability space obtained as the product of identical copies $(X_0, \mu_0)_k$ of (X_0, μ_0) , $k \in K$. Let $\sigma : \Gamma \rightarrow \text{Aut}(X, \mu)$ be defined by $\sigma(g)((x_k)_k) = (x'_k)_k$, where $x'_k = x_{g^{-1}k}$. We call σ the (X_0, μ_0) -Bernoulli $(\Gamma \curvearrowright K)$ -action. We generically refer to such actions as *generalized Bernoulli actions*.

In case $K = \Gamma$ and $\Gamma \curvearrowright \Gamma$ is the left multiplication, we simply call σ the (X_0, μ_0) -Bernoulli Γ -action.

Note that if we denote $(A_0, \tau_0) = (L^\infty X_0, \int \cdot d\mu_0)$, then the algebra $(L^\infty X, \int \cdot d\mu)$ coincides with $\overline{\otimes}_k(A_0, \tau_0)$, with the action implemented by σ on elements of the form $\otimes_k a_k \in \overline{\otimes}_k(A_0, \tau_0)$ being given by $\sigma_g(\otimes_k a_k) = \otimes_k a'_k$, $a'_k = a_{g^{-1}k}$, $k \in K$, $g \in \Gamma$.

2.2.1. Lemma. 1° . *If either (X_0, μ_0) has no atoms and for all $g \neq e$ there exists $k \in K$ such that $gk \neq k$, or if (X_0, μ_0) is arbitrary and for all $g \neq e$ the set $\{k \in K \mid gk \neq k\}$ is infinite, then σ is a free action.*

2° . *σ is weakly mixing iff $\forall K_0 \subset K \exists g \in \Gamma$ such that $gK_0 \cap K_0 = \emptyset$ and iff any orbit of $\Gamma \curvearrowright K$ is infinite.*

3° . *σ is strongly mixing iff $\forall K_0 \subset K$ finite $\exists F \subset \Gamma$ finite such that $gK_0 \cap K_0 = \emptyset$, $\forall g \in \Gamma \setminus F$, and iff the stabilizer $\{h \in \Gamma \mid hk = k\}$ of any $k \in K$ is finite.*

Proof. Part 3° and the first equivalence in 2° are trivial (exercise!). To prove 2° note that if $K_0 \subset K$ is a finite set such that $gK_0 \cap K_0 \neq \emptyset$ then $\text{tr}(g\chi_{K_0}\chi_{K_0}) \geq |K_0|^{-1}$, $\forall g \in \Gamma$, where tr is the “trace” (or integral) on $\ell^\infty K$, corresponding to the measure giving mass 1 to each atom. Taking the element of minimal $\ell^2 K$ -norm in $\text{co}^w\{g\chi_{K_0} \mid g \in \Gamma\} \subset \ell^2(K)$, gives a Γ -invariant $b \in \ell^2(K)$ with $\text{tr}(b\chi_{K_0}) \neq 0$. But then any non-empty “level set” L_0 of b is finite and Γ -invariant (cf. Lemma 1.5.2). This ends the proof of 2° . We leave the proof of 1° as an exercise. Q.E.D.

2.2.2. Proposition. *Let Γ be a non-amenable group acting on a set K such that $\Gamma_k = \{g \in \Gamma \mid gk = k\}$ is amenable $\forall k \in K$. Then any (X_0, μ) -Bernoulli $(\Gamma \curvearrowright K)$ -action has spectral gap. In particular any Bernoulli Γ -action has spectral gap.*

Proof. Let $\{\eta_i\}_{i \in I_0}$ be an orthonormal basis for $L^2 X_0$. Denote by $(L^2 X_0)_k$, $k \in K$, copies of $L^2 X_0$ indexed by K and $\{\eta_i^k\}_i \subset (L^2 X_0)_k$ the corresponding copies of the orthonormal basis of $\{\eta_i\}_{i \in I_0}$. It is immediate to see that if we denote by \tilde{I} the set of multi-indices $(i_k)_{k \in K}$ with entries in I_0 such that $i_k = 0$ for all but finitely many i_k , then $E = \{\otimes_k \eta_{i_k}^k \mid (i_k)_k \in \tilde{I}\}$ is an orthonormal basis of the Hilbert space $L^2 X = \overline{\otimes}_k (L^2 X_0)_k$, where $(X, \mu) = \prod_k (X_0, \mu_0)_k$. Moreover, E is invariant to the action (or representation) σ of Γ on $L^2 X$ implemented by the Bernoulli $(\Gamma \curvearrowright K)$ -action. Thus, $\Gamma \curvearrowright L^2 X$ coincides with $\oplus_\eta \ell^2(\Gamma/\Gamma_\eta)$, where for $\eta \in E$ we denote $\Gamma_\eta = \{g \in \Gamma \mid \sigma_g(\eta) = \eta\}$, the stabilizer of η , and the direct sum is over the set \hat{E} of orbits of elements $\eta \in E$. If for $\eta \in E$ we denote $S_\eta \subset K$ the “support” of $\eta = \otimes_k \eta_{i_k}^k$, i.e. the set of $k \in K$ with $i_k \neq 0$, then S_η is a finite subset of K and each $g \in \Gamma_\eta$ leaves S_η invariant. Thus, the stabilizer of η , Γ_η , is contained in the stabilizer of the finite set $S_\eta \subset K$, $\Gamma_{S_\eta} = \{g \in \Gamma \mid gS_\eta = S_\eta\}$. But the latter has the amenable (by hypothesis) subgroup $\{g \in \Gamma \mid gk = k, \forall k \in S_\eta\}$ as a normal subgroup of finite index. Thus Γ_{S_η} is amenable implying that S_η is amenable and Lemma 2.1.3 applies. Q.E.D.

2.3. Gaussian actions. Let \mathcal{H}_n be the real Hilbert space of dimension $n \leq \infty$ and \mathcal{U}_n its orthogonal group (group of unitaries on \mathcal{H}_n). If $n < \infty$ then we view $\mathcal{H}_n \simeq \mathbb{R}^n$ as a probability space with the measure μ_n given by

$$(2\pi)^{-n/2} \int_{\mathcal{H}_n} \cdot e^{-\|t\|_2^2} d\lambda(t).$$

Note that if we view \mathcal{H}_n as \mathbb{R}^n then $(\mathcal{H}_n, \mu_n) = \prod_{i=1}^n (\mathcal{H}_1, \mu_1)_i$, where $\mathcal{H}_1 = \mathbb{R}$ and μ_1 is given by the Gaussian distribution on \mathbb{R} , i.e. $(2\pi)^{-1/2} \int_{\mathbb{R}} \cdot e^{-t^2/2} dt$.

Note that \mathcal{U}_n , with its Polish group topology (which is a compact group topology in fact) obviously acts (continuously) on (\mathcal{H}_n, μ_n) by μ_n -preserving transformations. The action is clearly free, because if $\alpha \in \mathcal{U}_n$ is non trivial then the fixed points of α is a proper vector subspace of \mathcal{H}_n , thus its measure in (\mathcal{H}_n, μ_n) is zero.

We then define on \mathcal{H}_∞ the infinite product probability space $(\mathcal{H}_\infty, \mu_\infty) = \prod_{i=1}^\infty (\mathbb{R}, \mu_1)$, or alternatively define $(L^\infty(\mathcal{H}_\infty, \mu_\infty), \tau_{\mu_\infty})$ as the von Neumann algebra inductive limit of $(L^\infty \mathcal{H}_n, \mu_n, \tau_{\mu_n})$. Then the infinite dimensional orthogonal group \mathcal{U}_∞ clearly acts μ_∞ -preservingly on \mathcal{H}_∞ as the closure of the $\cup_n \mathcal{U}_n$ -action (the embeddings $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ being given by say fixing an orthonormal basis for \mathcal{H}_∞ and identifying \mathcal{U}_n with the n -dimensional orthogonal group acting on the first n elements of the basis, while leaving all other elements in the basis fixed. It is easy to see that the action of \mathcal{U}_∞ is still free (Same argument works to show that any proper real Hilbert subspace of \mathcal{H}_∞ has μ_∞ -measure zero. Exercise!).

Any orthogonal representation π of a discrete group Γ on \mathcal{H}_n implements a free m.p. action $\sigma = \sigma^\pi$ of Γ on (\mathcal{H}_n, μ_n) , by composing π with the action of \mathcal{U}_n . It is an easy exercise to show that if $n = \infty$ and π is a weak mixing orthogonal representation of Γ on \mathcal{H}_∞ (i.e. without finite dimensional invariant subspaces) then the action σ is weak mixing (Exercise!). This also follows from the following:

2.3.1. Lemma. *If $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\infty)$ is a representation then, when viewed as a representation of Γ on $L^2(\mathcal{H}_\infty, \mu_\infty)$, σ^π is contained in $\oplus_{n \geq 0} \pi_{\mathbb{C}}^{\otimes n}$, where $\pi_{\mathbb{C}}$ is the complexification of π . More precisely, if we denote by $\rho^{\otimes_s n}$ the symmetric tensor product of a rep ρ then $\sigma^\pi = \oplus_{n \geq 0} \pi_{\mathbb{C}}^{\otimes_s n}$*

Proof. Dan's presentation.

2.3.2. Corollary. *Let $\pi : \Gamma \rightarrow \mathcal{H}_\infty$ be an orthogonal representation of the discrete group Γ . Then $1_\Gamma \not\prec \pi \Leftrightarrow \sigma^\pi$ has spectral gap $\Leftrightarrow \sigma^\pi$ is strongly ergodic.*

Proof. Immediate by 2.3.1.

Q.E.D.

2.4. Left actions from group embeddings. Let G be a locally compact group with λ its Haar measure. A countable subgroup $\Lambda \subset G$ is called a *lattice* in G if it admits a

fundamental domain, i.e. a measurable subset $S \subset G$ such that the sets $\{gS\}_{g \in \Lambda}$ are disjoint and $\cup_g gS = X$ a.e. Denote by μ the probability measure on the homogeneous space G/Λ given by the natural identification of G/Λ with S (modulo a set of measure zero), the latter being endowed with the measure $\lambda|_S$ (after renormalizing λ so that $\lambda(S) = 1$).

The left action of the group G on G/Λ is then clearly measure preserving. Thus, any discrete subgroup $\Gamma \subset G$ acts on $(G/\Lambda, \mu)$ by left translations.

A particular case of interest is when G is a compact group, λ its Haar measure normalized so that $\lambda(G) = 1$ and we take $\Lambda = \{e\}$. The action of G on itself by left translation is then clearly continuous and compact. It is also free, because if $h \in G \setminus \{e\}$ then the set $\{k \in G \mid hk \neq k\}$ is empty.

Take now $\Gamma \subset G$ a dense subgroup and consider the action by left translation $\Gamma \curvearrowright G$, i.e. the restriction to Γ of $G \curvearrowright G$.

2.4.1. Lemma. *The action $\Gamma \curvearrowright G$ is free and ergodic.*

Proof. We have already shown that the action is free. To see it is ergodic note first that if $f \in L^1(G, \lambda)$ is Γ -invariant then for any $f' \in L^1G$ the map $f * f'(k) = \int f(kg^{-1})f'_g d\lambda(g)$ is still Γ -invariant. If we now take $f_n \in L^1G$ to be an approximate identity for the algebra $(L^1G, *)$ then $f * f_n$ is Γ -invariant and continuous, thus $f * f_n$ is constant. But $\|f - f * f_n\|_1 \rightarrow 0$, thus f is constant as well. Q.E.D.

2.5. Quotients and products. Given an action $\Gamma \curvearrowright^\sigma (X, \mu)$ (of which we have plenty of examples by now), let $A \subset L^\infty X$ be a σ -invariant von Neumann subalgebra, i.e. a weakly closed $*$ -subalgebra containing 1 and such that $\sigma_g(A) = A$, $\forall g \in \Gamma$. By Theorem 1.1, $(A, \tau_\mu) = (L^\infty(Y, \nu), \tau_\nu)$, for some standard probability space (Y, ν) , with the restriction $\theta_g = \sigma_{g|_A}$, $g \in \Gamma$, implementing an action θ of Γ on $L^\infty Y$, thus coming from an action $\Gamma \curvearrowright^\theta (Y, \nu)$. Such actions are called *quotients* of σ and can be quite useful. Properties such as ergodicity, (weak) mixing, strong ergodicity and spectral gap are clearly inherited by θ , but not freeness (in general).

We'll also often need to take the (diagonal) product of finitely or infinitely (but countably) many actions σ_i of the same group Γ on (X_i, μ_i) , $i = 1, 2, \dots$, thus getting an action $\sigma = \sigma_1 \times \sigma_2 \times \dots$ of Γ on $(X, \mu) = \prod_i (X_i, \mu_i)$. On function spaces, this corresponds to the (diagonal tensor) product action $\sigma = \otimes_i \sigma_i$ of Γ on $\overline{\otimes}_i (L^\infty X_i, \tau_{\mu_i})$.

It is easy to see that the product of a properly outer transformation with any other transformation is still properly outer (exercise). Thus, if $\Gamma \curvearrowright^\sigma X$ is free then $\sigma_g \otimes \rho_g$, $g \in \Gamma$ is free for any other action $\Gamma \curvearrowright^\rho Y$. By Lemma 1.3.1 if all σ_i are weak mixing Γ -actions then $\otimes_i \sigma_i$ is weak mixing.

If ρ is not strongly ergodic, then $\sigma \otimes \rho$ is not strongly ergodic $\forall \sigma$. If σ has spectral gap (resp. is strongly ergodic) then any asymptotically invariant (resp. non-trivial as. inv.) sequence for $\sigma \otimes \rho$ is asymptotically contained in $L^\infty Y$ (exercise).

The following combination of Bernoulli shifts and products of actions will be of interest to us: Let σ_0 be an action of Γ_0 on (X_0, μ_0) . Let also Γ_1 be another discrete group and α an action of Γ_1 on Γ_0 by group automorphisms. (N.B.: The action α may be trivial.) Let σ_1 be the Bernoulli shift action of Γ_1 on $(X, \mu) = \overline{\otimes}_{g_1 \in \Gamma_1} (X_0, \mu_0)_{g_1}$. Let also σ_0^γ be the action of Γ_0 on (X, μ) given by $\sigma_0^\alpha = \otimes_{g_1} \sigma_0 \circ \alpha(g_1)$.

2.5.1. Lemma. 1°. *We have $\sigma_1(g_1)\sigma_0^\alpha(g_0)\sigma_1(g_1^{-1}) = \sigma_0^\alpha(\alpha(g_1)(g_0))$, for any $g_0 \in \Gamma_0$ and $g_1 \in \Gamma_1$. Thus, $(g_0, g_1) \mapsto \sigma_0^\alpha(g_0)\sigma_1(g_1)$ implements an action $\sigma = \sigma_0 \rtimes_\alpha \sigma_1$ of $\Gamma_0 \rtimes_\alpha \Gamma_1$ on (B, τ) .*

2°. *If the group Γ_0 is infinite and the action σ_0 is properly outer then the action σ defined in 1° is properly outer.*

3°. *If the action σ_0 is weakly mixing, or if the group Γ_1 is infinite, then σ is weakly mixing (thus ergodic).*

4°. *If the group Γ_1 is non-amenable, then σ has spectral gap.*

Proof. 1° is straightforward direct calculation.

2° follows once we notice that if Γ_0 is infinite and σ_0 is properly outer, it automatically follows that X_0 has no atomic part. This in turn implies the Bernoulli shift of Γ_1 on $(X_0, \mu_0)^{\otimes \Gamma_1}$ is a properly outer action, even when Γ_1 is a finite group.

3°. This follows by the observations at the beginning of 2.4.

4°. This follows from part 2° of Corollary 2.1.4.

Q.E.D.

3. FULL (PSEUDO)GROUPS AND ORBIT EQUIVALENCE OF ACTIONS

3.1. Full groups and pseudogroups. Let $\mathcal{G} \subset \text{Aut}(X, \mu)$ be a (discrete) subgroup of automorphisms of the probability space (X, μ) . Following H. Dye ([Dy1]), \mathcal{G} is called a *full group* if the following implication holds true: If $\phi \in \text{Aut}(X, \mu)$ is so that there exists a countable partition of X into measurable subsets $X_n \subset X$ and $\phi_n \in \mathcal{G}$ such that $\phi|_{X_n} = \phi_n|_{X_n}$ must be contained in \mathcal{G} . (Note that this condition automatically entails that $\phi_n(X_n)$ is also a partition of X .) In other words, any automorphism of (X, μ) which coincides “piecewise” with automorphisms in \mathcal{G} must lie itself in \mathcal{G} . An isomorphism of full groups \mathcal{G}_i on (X_i, μ_i) , $i = 1, 2$, is an isomorphism $\Delta : (X_1, \mu_1) \simeq (X_2, \mu_2)$ satisfying $\Delta \mathcal{G}_1 \Delta^{-1} = \mathcal{G}_2$.

The next two results are from ([Dy1]).

3.1.1. Lemma. *Let $S \subset \text{Aut}(X, \mu)$ and let \mathcal{S} denote the subgroup of $\text{Aut}(X, \mu)$ generated by S . Denote $[S]$ the set of automorphisms $\phi \in \text{Aut}(X, \mu)$ for which there exist a partition of X , $\{X_n\}_n$ and $\phi_n \in \mathcal{S}$, such that $\phi|_{X_n} = \phi_n|_{X_n}$. Then $[S]$ is a full group and it is in fact the smallest full group that contains S .*

Proof. Trivial by the definitions.

Q.E.D.

The argument in the next lemma is reminiscent of the Murray-von Neumann proof that if a von Neumann algebra M is finite (i.e. $u \in M$, $uu^* = 1$ implies $u^*u = 1$), then any partial isometry $v \in M$ can be extended to a unitary in M .

3.1.2. Lemma. *Let \mathcal{G} be a full group. If $X_1, X_2, \dots \subset X$ are disjoint measurable subsets and $\phi_1, \phi_2, \dots \in \mathcal{G}$ are so that $\phi_n(X_n)$ are disjoint then there exists $\phi \in \mathcal{G}$ such that $\phi|_{X_n} = \phi_n|_{X_n}, \forall n$.*

Proof. Denote $R = \cup_{n \geq 1} X_n$, $L = \cup_{n \geq 1} \phi_n(X_n)$ and let $\psi : R \simeq L$ be the isomorphism given by $\psi|_{X_n} = \phi_n|_{X_n}$. Let also $Y_0 = X \setminus R$.

It is clearly sufficient (by maximality argument) to prove that there exists $\phi_0 \in \mathcal{G}$ such that $\phi_0(Y_0) \cap (X \setminus L) \neq \emptyset$ (a.e.). Assume this is not the case and denote by Y'_0 the \mathcal{G} -centralizer of Y_0 . This is defined a.e. as follows: For each $\phi \in \mathcal{G}$ let $p_\phi = \chi_{\phi(Y_0)}$ viewed as an element in $L^\infty X$. Then let $p = \vee \{p_\phi \mid \phi \in \mathcal{G}\}$ and choose $Y'_0 \subset X$ so that $\chi_{Y'_0} = p$ in $L^\infty X$.

By the definition of Y'_0 , the contradiction assumption implies Y'_0 is disjoint from $X \setminus L$, thus $Y'_0 \subset L$, so in particular $Y_0 \subset L$. This implies there exists $Y_1 \subset R$ such that $\psi(Y_1) = Y_0$. Since $Y_0 \subset Y'_0$ we also have $Y_1 \subset Y'_0$. Assume we constructed disjoint subsets Y_1, Y_2, \dots, Y_k of R such that $\psi(Y_{i+1}) = Y_i, i = 0, 1, \dots, k-1$, with $Y_i \subset Y'_0, \forall 0 \leq i \leq k$. In particular, since $Y'_0 \subset L$, we have $Y_k \subset L$, implying there exists $Y_{k+1} \subset R$ such that $\psi(Y_{k+1}) = Y_k$. Also, for all $0 \leq i \leq k-1$ we have $\psi(Y_{k+1} \cap Y_{i+1}) = \psi(Y_{k+1}) \cap \psi(Y_{i+1}) = Y_k \cap Y_i = \emptyset$. We also have $Y_{k+1} \cap Y_0 \subset R \cap Y_0 = \emptyset$.

This means we can construct recursively a whole sequence of disjoint subsets $\{Y_k\}_{k \geq 0}$ such that $\psi(Y_{k+1}) = Y_k, \forall k \geq 0$. In particular, all Y_k have the same measure, implying $\mu(\cup_k Y_k) = \infty$, a contradiction. Q.E.D.

A measurable, measure preserving a.e. isomorphism $\phi : R \rightarrow L$, for $R, L \subset X$ measurable with $\mu(L) = \mu(R)$, $\mu|_L \circ \phi = \mu|_R$ will be called a *local isomorphism* of (X, μ) . The sets R, L are called the *right*, resp. *left supports* of ϕ . We denote by 0 the local isomorphism with empty (a.e.) left-right supports. The *composition* $\psi\phi$ of two local isomorphisms ϕ, ψ is by definition the local isomorphism with right support $R(\psi\phi) = \{t \in R(\phi) \mid \phi(t) \in R(\psi)\}$ acting by $\psi\phi(t) = \psi(\phi(t)), t \in R(\psi\phi)$, and of course left support equal to $\{\psi\phi(t) \mid t \in R(\psi\phi)\}$. The inverse ϕ^{-1} of a local isomorphism ϕ is the local isomorphism with right support $R(\phi^{-1}) = L(\phi)$ defined on this set as the inverse of ϕ . We make the convention that $0^{-1} = 0$.

A set \mathcal{G}^p of local isomorphisms of (X, μ) is a *pseudogroup* if it contains $0, 1_X$ and is closed to composition and inverse operations. It is a *full pseudogroup* if it is a pseudogroup and satisfies the following conditions:

(3.1.i). If $\phi \in \mathcal{G}^p$ and $Y \subset R(\phi)$ is measurable then $\phi|_Y \in \mathcal{G}^p$.

(3.1.ii). If ϕ is a local isomorphism such that there exists a countable partition of $R(\phi)$ with measurable subsets $\{R_n\}_n$ with the property that $\phi|_{R_n} \in \mathcal{G}^p, \forall n$, then $\phi \in \mathcal{G}^p$.

Conditions (i) and (ii) state that \mathcal{G}^p is closed to “cutting” (restrictions) and countable “pasting”. Like for full groups, an isomorphism of full pseudogroups is an isomorphism of probability spaces taking one full pseudogroup onto the other.

3.1.3. Lemma. 1°. Given a set S of local isomorphisms of (X, μ) , denote by \mathcal{S} the pseudogroup generated by S and by $[S]^p$ the set of local isomorphisms ϕ with the property that there exist a partition of $R(\phi)$, $\{X_n\}_n$, and $\phi_n \in \mathcal{S}$, with $X_n \subset R(\phi_n)$, such that $\phi|_{X_n} = \phi_n|_{X_n}$. Then $[S]^p$ is a full pseudogroup and it is in fact the smallest full pseudogroup that contains S .

2°. If \mathcal{G}^p is a full pseudogroup then the set $\mathcal{G}_1^p = \{\phi \in \mathcal{G}^p \mid R(\phi) = X\}$ is a full group, called the full group associated with \mathcal{G}^p .

3°. If \mathcal{G} is a full group then the set $\mathcal{G}^p = \{\phi|_Y \mid \phi \in \mathcal{G}, Y \subset X\}$ is a full pseudogroup called the pseudogroup associated with \mathcal{G} . With the notations in 2° we have $(\mathcal{G}^p)_1 = \mathcal{G}$.

Proof. Trivial by 3.1.2.

Q.E.D.

We denote $\mathcal{G}_0^p = \{\phi \in \mathcal{G}^p \mid \psi\phi = \psi, \forall \psi, R(\psi) = L(\phi)\}$ the set of *units* of \mathcal{G}^p . Note that it can be naturally identified with the lattice of projections of $L^\infty X$ (after identifying the local iso differing on a set of measure 0). We denote $\mathcal{Z}(\mathcal{G}^p) = \{\phi \in \mathcal{G}^p \mid \phi\psi = \psi\phi, \forall \psi \in \mathcal{G}^p\}$ the *centralizer* of \mathcal{G}^p .

A full group \mathcal{G} is *ergodic* if its action on (X, μ) is ergodic. A full pseudogroup \mathcal{G}^p is ergodic if its associated full group is ergodic. We have:

3.1.4. Lemma. 1° . $\mathcal{Z}(\mathcal{G}^p) \subset \mathcal{G}_0^p$ and \mathcal{G}^p is ergodic iff $\mathcal{Z}(\mathcal{G}^p) = \{1\}$.

2° . If \mathcal{G} (resp. \mathcal{G}^p) is ergodic then for any subsets $R, L \subset X$ with $\mu(R) = \mu(L)$ there exists $\phi \in \mathcal{G}$ (resp. $\phi \in \mathcal{G}^p$) such that $\phi(R) = L$ (resp. $R(\phi) = R, L(\phi) = L$).

Proof. Part 1° is left as an exercise. The two conditions in 2° clearly imply one another and the one referring to full pseudogroups is trivial by a maximality argument.

Q.E.D.

Let us finally mention that any *algebraic isomorphism* of full pseudogroups \mathcal{G}_i^p on (X_i, μ_i) , $i = 1, 2$, i.e. a bijective map $\alpha : \mathcal{G}_1^p \rightarrow \mathcal{G}_2^p$ which preserves the product comes from an isomorphism of full pseudogroups, i.e. there exists $\Delta : (X_1, \mu_1) \simeq (X_2, \mu_2)$ such that $\alpha(\phi)\Delta = \Delta\phi$, $\forall \phi \in \mathcal{G}_1^p$. This is easy to prove, and we leave it as an exercise. It is less trivial to show that in fact a similar statement holds true for full groups as well:

3.1.5. Theorem [Dye 59]. If \mathcal{G}_i is a full pseudogroup on (X_i, μ_i) , $i = 1, 2$, and $\alpha : \mathcal{G}_1 \simeq \mathcal{G}_2$ is an (algebraic, plain) isomorphism of groups, then there exists $\Delta : (X_1, \mu_1) \simeq (X_2, \mu_2)$ such that $\alpha(\phi) = \Delta\phi\Delta^{-1}$, $\forall \phi \in \mathcal{G}_1$.

Proof. Presented by Julien.

Q.E.D.

3.2. Orbit equivalence of group actions. Following H. Dye ([Dy1]), two subgroups of automorphisms $\Gamma \subset \text{Aut}(X, \mu)$ and $\Lambda \subset \text{Aut}(Y, \nu)$ are *weakly equivalent* if there exists an isomorphism $\Delta : (X, \mu) \simeq (Y, \nu)$ such that $\Delta([\Gamma])\Delta^{-1} = [\Lambda]$. Note that by Lemma 3.1.3 this is equivalent to the fact that $\Delta([\Gamma]^p) = [\Lambda]^p$.

Two (faithful) actions $\Gamma \curvearrowright^\sigma (X, \mu)$ and $\Lambda \curvearrowright^\theta (Y, \nu)$ are *weakly equivalent* if $\sigma(\Gamma), \theta(\Lambda)$ are weakly equivalent. The isomorphism Δ is called a *weak equivalence* of the corresponding automorphism groups, or actions.

Following Feldman-Moore ([FM77]), two groups of automorphisms $\Gamma \subset \text{Aut}(X, \mu)$, $\Lambda \subset \text{Aut}(Y, \nu)$ are *orbit equivalent* (abbreviated *OE*) if there exists $\Delta : (X, \mu) \simeq (Y, \nu)$ and a set $X_0 \subset X$ of measure zero such that $\Delta(\Gamma t) = \Lambda(\Delta t)$, $\forall t \in X \setminus X_0$. Two (faithful) actions $\Gamma \curvearrowright^\sigma (X, \mu)$ and $\Lambda \curvearrowright^\theta (Y, \nu)$ are *orbit equivalent* if $\sigma(\Gamma), \theta(\Lambda)$ are orbit equivalent. We then write $\sigma \sim_{OE} \theta$, or $(\Gamma \curvearrowright X) \sim_{OE} (\Lambda \curvearrowright Y)$. An isomorphism Δ satisfying such condition is called an *orbit equivalence* of the corresponding automorphism groups (or actions), and we write $\sigma \stackrel{\Delta}{\sim}_{OE} \theta$ when we want to emphasize it.

The next lemma, due to Feldman and Moore, relates the orbit and weak equivalence, showing they are “the same”.

3.2.1. Lemma. Let $\Gamma \subset \text{Aut}(X, \mu)$, be a countable group.

1° . If $\Gamma' \subset [\Gamma]$ is any other countable subgroup that generates $[\Gamma]$ as a full group, i.e. $[\Gamma'] = [\Gamma]$, then for almost all $t \in X$ we have $\Gamma t = \Gamma' t$.

2° . Let \mathcal{R}_Γ be the equivalence relation given by the orbits of Γ . If $\phi \in \text{Aut}(X, \mu)$ then $\phi \in [\Gamma]$ iff the graph of ϕ is contained in \mathcal{R}_Γ . Same for $[\Gamma]^p$.

3°. If $\Lambda \subset \text{Aut}(Y, \nu)$ is a countable group and $\Delta : (X, \mu) \simeq (Y, \nu)$ is an isomorphism, then Δ is an orbit equivalence of Γ, Λ if and only if it is a weak equivalence.

Proof. Parts 1°, 2° and the implication \Leftarrow of 3° are trivial. Proving \Rightarrow in 3° amounts to show that if $\phi : (X, \mu) \simeq (Y, \nu)$ satisfies $\phi(t) \in \Gamma t, \forall t \in R$, then $\phi \in [\Gamma]$. Fixing $g \in \Gamma$, note that the set $X_g = \{t \in R \mid \phi(t) = gt\}$ (i.e. the set X_g on which ϕ coincides with g) is measurable and by hypothesis we have $\cup_g X_g = R$. But this means $\phi \in [\Gamma]$. Q.E.D.

The above lemma shows that the full group (or pseudogroup) $[\Gamma]$ of a countable group $\Gamma \subset \text{Aut}(X, \mu)$ is completely encoded (up to weak equivalence) by the equivalence relation $\mathcal{R}_\Gamma = \{(t, gt) \mid t \in \Gamma\} \subset X \times X$ (up to OE). Note that \mathcal{R}_Γ is well defined only up to a set $t \in X$ of measure 0, but that this is not a problem since the \mathcal{R}_Γ saturated of any $X_0 \subset X$ of measure 0 has measure zero.

A *standard, measurable, measure preserving, countable equivalence relation* (hereafter called a *standard equivalence relation*) on (X, μ) is an equivalence relation \mathcal{R} on X with the property that: (a) Each orbit of \mathcal{R} is countable; (b) The \mathcal{R} -saturated of any subset $X_0 \subset X$ of measure 0 has measure zero; (c) There exists a countable subgroup $\Gamma \subset \text{Aut}(X, \mu)$ such that for almost all $t \in X$ the orbit of t under \mathcal{R} coincides with Γt .

Related to this abstract notion of equivalence relation, it is convenient to say that a full group \mathcal{G} (resp. full pseudogroup \mathcal{G}^p) is *countably generated* if there exists $\exists S \subset \mathcal{G}$ (resp. $S \subset \mathcal{G}^p$) at most countable such that $[S] = \mathcal{G}$ (resp. $[S]^p = \mathcal{G}^p$). It is trivial to see that a full group \mathcal{G} is countably generated iff its associated pseudogroup \mathcal{G}^p is countably generated.

From the above considerations we see that a countably generated full group \mathcal{G} is “same as” the standard, measurable, measure preserving, countable equivalence relation \mathcal{R} implemented by any of the countable subgroups Γ of \mathcal{G} that generates \mathcal{G} .

Let us finally mention that the abstraction of countable equivalence relations can be pushed a bit further. Thus, it is shown in ([FM]) that if one considers a standard Borel structure \mathcal{X} underlying (X, μ) , then any countable equivalence relation \mathcal{R} as above comes from an equivalence relation on $X \times X$ with the property that \mathcal{R} lies in the product Borel structure $\mathcal{X} \times \mathcal{X}$, with each orbit of \mathcal{R} countable, and generated by local m.p. Borel maps between Borel subsets of X with graph included into \mathcal{R} .

3.3. Amplifications and stable OE. If \mathcal{G} (resp. \mathcal{G}^p) is a full group (resp. pseudogroup) and $Y \subset X$ then we denote by \mathcal{G}_Y (resp. \mathcal{G}_Y^p) the set of automorphisms (resp. local isomorphism) ψ of (Y, μ_Y) for which there exists $\phi \in \mathcal{G}$ (resp. $\phi \in \mathcal{G}^p$) with $\phi|_Y = \psi$. We call \mathcal{G}_Y (resp. \mathcal{G}_Y^p) the *restriction* of \mathcal{G} (resp. \mathcal{G}^p) to Y .

3.3.1. Lemma. *Let \mathcal{G} be a full group on (X, μ) .*

1°. *If $Y \subset X$ is a measurable subset of non-zero measure then \mathcal{G}_Y (resp. \mathcal{G}_Y^p) is a full group (resp. pseudogroup) on (Y, μ_Y) and we have $(\mathcal{G}_Y)^p = (\mathcal{G}^p)_Y$.*

2°. If \mathcal{G} is countably generated then so is \mathcal{G}_Y . Also, if \mathcal{R} is the equivalence relation implemented by \mathcal{G} then the equivalence relation associated with \mathcal{G}_Y is equal to $\mathcal{R}_Y \stackrel{\text{def}}{=} \mathcal{R} \cap Y \times Y$. Thus, if $\Gamma \subset \text{Aut}(X, \mu)$ is countable and such that $\mathcal{R} = \mathcal{R}_\Gamma$, then the orbits of \mathcal{R}_Y are given by $\Gamma t \cap Y$, for almost all $t \in Y$.

3°. If $Y_1, Y_2 \subset X$ are so that there exists a local isomorphism $\phi \in \mathcal{G}^p$ such that $R(\phi) = Y_1$, $L(\phi) = Y_2$ then ϕ implements an isomorphism from \mathcal{G}_{Y_1} onto \mathcal{G}_{Y_2} (as well as between the corresponding full pseudogroups). In particular, if \mathcal{G} is ergodic then for any $Y_1, Y_2 \subset X$ with $\mu(Y_1) = \mu(Y_2) > 0$ we have $\mathcal{G}_{Y_1} \simeq \mathcal{G}_{Y_2}$ and $\mathcal{G}_{Y_1}^p \simeq \mathcal{G}_{Y_2}^p$.

Proof. This is trivial by the definitions and 3.1.4.

Q.E.D.

If $n \geq 1$ then denote by \mathcal{G}^n (resp. $\mathcal{G}^p{}^n$) the full group (resp. full pseudogroup) generated on the product of (X, μ) and $(\{1, \dots, n\}$ with the counting measure by $\mathcal{G} \times id$ and the permutations of $\{1, \dots, n\}$. These full (pseudo)groups are clearly ergodic. For each $t > 0$ we denote by \mathcal{G}^t (resp. $(\mathcal{G}^p)^t$) the isomorphism class of \mathcal{G}_Y (resp of \mathcal{G}_Y^p) where $n \geq t$ and $Y \subset X \times \{1, \dots, n\}$ is a subset of measure t/n . This clearly doesn't depend on n and Y .

The full group \mathcal{G}^t (resp. full pseudogroup $(\mathcal{G}^p)^t$) is called the *amplification by t of \mathcal{G}* (resp. of \mathcal{G}^p).

Two ergodic actions $\Gamma \curvearrowright^\sigma X, \Lambda \curvearrowright^\theta Y$ are *stably orbit equivalent* (stably OE) if there exists subsets $X_0 \subset X, Y_0 \subset Y$ of positive measure such that $[\sigma(\Gamma)]_{Y_0} \simeq [\theta(\Lambda)]_{Y_0}$. Note that this condition holds true iff $[\sigma(\Gamma)] \simeq [\theta(\Lambda)]^t$, where $t = \nu(Y_0)/\mu(X_0)$. We write $\sigma \simeq_{OE_t} \theta$. The constant t is called the *coupling constant* (or *amplification constant*) of the stable OE.

4. VON NEUMANN ALGEBRAS FROM GROUP ACTIONS

4.1. The group measure space construction. A key tool in the study of actions is the so-called *group measure space construction* of Murray and von Neumann ([MvN1]), which associates to $\Gamma \curvearrowright (X, \mu)$ the von Neumann algebra $L^\infty X \rtimes \Gamma$ generated on the Hilbert space $\mathcal{H} = L^2 X \overline{\otimes} \ell^2 \Gamma$ by a copy of the algebra $L^\infty X$, acting on \mathcal{H} by left multiplication on the component $L^2 X$ of the tensor product $L^2 X \overline{\otimes} \ell^2 \Gamma$, and a copy of the group Γ , acting on \mathcal{H} as the multiple of left regular representation given by the unitary operators $u_g = \sigma_g \otimes \lambda_g$, $g \in \Gamma$, where $\sigma_g, g \in \Gamma$, is viewed here as a unitary representation on $L^2 X$.

The following more concrete description of $M = L^\infty X \rtimes \Gamma$ and its standard representation is quite useful: Identify $\mathcal{H} = L^2 X \overline{\otimes} \ell^2 \Gamma$ with the Hilbert space of ℓ^2 -summable formal sums $\sum_g \xi_g u_g$, with “coefficients” ξ_g in $L^2 X$ and “undeterminates” $\{u_g\}_g$ labeled by the elements of the group Γ . Define a $*$ -operation on \mathcal{H} by $(\sum_g \xi_g u_g)^* = \sum_g \sigma_g(\xi_{g^{-1}}^*) u_g$ and let both $L^\infty X$ and the u_g ’s act on \mathcal{H} by left multiplication, subject to the product rules $y(\xi u_g) = (y\xi)u_g$, $u_g(\xi u_h) = \sigma_g(\xi)u_{gh}$, $\forall g, h \in G$, $y \in L^\infty X$, $\xi \in L^2 X$. In fact, given any $\xi = \sum_g \xi_g u_g, \zeta = \sum_h \zeta_h u_h \in \mathcal{H}$ one can define the product $\xi \cdot \zeta$ as the formal sum $\sum_k \eta_k u_k$ with coefficients $\eta_k = \sum_g \xi_g \zeta_{g^{-1}k}$, the sum being absolutely convergent in the norm $\|\cdot\|_1$ on $L^1 X$, with estimates $\|\eta_k\|_1 \leq \|\xi\|_2 \|\zeta\|_2$, $\forall k \in \Gamma$, by the Cauchy-Schwartz inequality. In other words, $\xi \eta \in \ell^\infty(\Gamma, L^1 X) \supset \ell^2(\Gamma, L^2 X) = \mathcal{H}$.

We say that $\xi \in \mathcal{H}$ is a *convolver* if $\xi \zeta \in \mathcal{H}$ (i.e. with the above notations $\eta_k \in L^2 X$ and $\sum_k \|\eta_k\|_2^2 < \infty$) for all $\zeta \in \mathcal{H}$. By the closed graph theorem it follows that $L_\xi(\zeta) = \xi \zeta$, $\zeta \in \mathcal{H}$, defines a linear bounded operator on \mathcal{H} . It is immediate to see that L_ξ^* then coincides with L_{ξ^*} , showing that the set of convolvers is closed to the $*$ -operation.

Then $M = L^\infty X \rtimes \Gamma$ in its standard representation on $L^2 M$ is nothing but the algebra of all left multiplication operators L_ξ by convolvers ξ . Its commutant in $\mathcal{B}(\mathcal{H})$ is the algebra of all right multiplication operators $R_\xi(\zeta) = \zeta \xi$, by convolvers ξ . If $T \in M$ then $\xi = T(1) \in \mathcal{H}$ is a convolver and T is the operator of left multiplication by ξ . The left multiplication by convolvers supported on $L^\infty X = L^\infty X u_e$ give rise to the (multiple of the) standard representation of $L^\infty X$, while the left multiplication by the convolvers $\{u_g\}_g$ give rise to the copy of the left regular representation of Γ . The integral τ_μ on $L^\infty X$ extends to a trace on $L^\infty X \rtimes \Gamma$ by $\tau(\sum_g y_g u_g) = \tau_\mu(y_e) = \langle \xi, 1 \rangle = \langle \xi \cdot 1, 1 \rangle$, where $\xi = \sum_g y_g u_g$. The Hilbert space \mathcal{H} naturally identifies with $L^2(M, \tau)$, with M as a subspace of $L^2 M$ identifying with the set of convolvers and the standard representation of M as left multiplication by convolvers.

All one has to retain from the above construction is: any element in $M = L^\infty X \rtimes \Gamma$ has a Fourier expansion; the way such Fourier expansions multiply; a Fourier expansion $x = \sum_g a_g u_g$ of an element in M , as opposed to an arbitrary square summable vector $\xi = \sum_g \xi_g u_g \in \mathcal{H} = \oplus_g L^2 X u_g$, has the property that the multiplication by x of any square summable $\eta \in \mathcal{H}$ stays in \mathcal{H} .

The following “twisted” version of the above construction is quite useful: In addition to $\Gamma \curvearrowright (X, \mu)$, take now a (normalized) $\mathcal{U}(X)$ -valued 2-cocycle for (σ, Γ) , i.e. a map $v : \Gamma \times \Gamma \rightarrow \mathcal{U}(X)$ such that $v_{g,h}v_{gh,k} = \sigma_g(v_{h,k})v_{g,hk}$, $\forall g, h, k \in \Gamma$, $v_{g,e} = v_{e,g} = 1$. On the same Hilbert space $\mathcal{H} = \oplus_g L^2 X u_g$ as before, consider a new product by $a(\xi_g u_g) = (a\xi_g)u_g$ and $u_h(\xi_g u_g) = \sigma_h(\xi_g)v_{h,g}u_{hg}$, then follow exactly the same procedure as above to get a von Neumann algebra of (left multiplication operators by) convolvers $L^\infty X \rtimes_{\sigma,v} \Gamma$. The formula for the trace is the same.

We say that two cocycles v, v' for σ are *equivalent* (or *cohomologous*), and we write $w' \sim w$, if there exists $w : \Gamma \rightarrow \mathcal{U}(X)$ such that $v'_{g,h} = w_g \sigma_g(w_h) v_{g,h} w_{gh}^*$, $\forall g, h \in \Gamma$. It is immediate to see that if $v' \sim v$ then the unitary operator U on $\mathcal{H} = \Sigma_g L^2 X u_g$ defined by $U(\Sigma_g \xi_g u_g) = \Sigma_g \xi_g w_g u_g$ implements a spatial isomorphism taking $L^\infty X \rtimes_{\sigma,v} \Gamma$ onto $L^\infty X \rtimes_{\sigma,v'} \Gamma$, more precisely U is $L^\infty X$ bimodular and satisfies $U u_g U^* = u'_g$.

4.1.1. Theorem. *Let $\Gamma \curvearrowright^\sigma (X, \mu)$ be a group action and v a 2-cocycle for (σ, Γ) .*

1°. *$L^\infty X$ is maximal abelian in $L^\infty X \rtimes_{\sigma,v} \Gamma$ iff $\Gamma \curvearrowright X$ is free.*

2°. *If $\Gamma \curvearrowright X$ is free then $L^\infty X \rtimes_{\sigma,v} \Gamma$ is a factor iff $\Gamma \curvearrowright X$ is ergodic.*

4.2. The von Neumann algebra of a full pseudogroup. Let now \mathcal{G}^p be a given full pseudogroup on (X, μ) . Then let $\mathbb{C}\mathcal{G}^p$ denote the algebra of formal finite linear combinations $\sum \phi c_\phi u_\phi$. Let $\tau(u_\phi)$ denote the μ -measure of the largest set on which ϕ acts as the identity and extend it by linearity to $\mathbb{C}\mathcal{G}^p$. Then define a sesquilinear form on $\mathbb{C}\mathcal{G}^p$ by $\langle x, y \rangle = \tau(y^*x)$ and denote by $L^2(\mathcal{G}^p)$ the Hilbert space obtained by completing $\mathbb{C}\mathcal{G}^p/I_\tau$ in the norm $\|x\|_2 = \tau(x^*x)^{1/2}$, where $I_\tau = \{x \mid \langle x, x \rangle = 0\}$. Each $\phi \in \mathcal{G}^p$ acts on $L^2(\mathcal{G}^p)$ as the left multiplication operator by u_ϕ , given by $u_\phi(u_\psi) = u_{\phi\psi}$. Denote by $L(\mathcal{G}^p)$ the von Neumann algebra generated by the operators $\{u_\phi, \phi \in \mathcal{G}^p\}$ and by $L(\mathcal{G}_0^p) \simeq L^\infty(X, \mu)$ the von Neumann subalgebra generated by the units \mathcal{G}_0^p .

Note that in the above we could equally start with the $*$ -algebra $M_0 = A\mathcal{G}^p$ of finite formal sums $\sigma_\phi a_\phi u_\phi$ subject to multiplication rule $(au_\phi)(bu_\psi) = a\phi(b\chi_{R(\phi)})u_{\phi\psi}$ and trace $\tau(au_\phi) = \int a\chi_{R(\phi)}d\mu$, the resulting Hilbert space $L^2(\mathcal{G}^p)$ and von Neumann algebra $L(\mathcal{G}^p)$, obtained by taking the weak closure of the algebra of left multiplication operators by elements in M_0 on $L^2(\mathcal{G}^p)$, being the same as when starting with $M_0 = \mathbb{C}\mathcal{G}^p$.

It is easy to check that $L(\mathcal{G}^p)$ is a finite von Neumann algebra, with the subalgebra $L(\mathcal{G}_0^p) = L^\infty(X, \mu)$ being maximal abelian in it, and that the vector state $\tau = \langle \cdot, 1 \rangle$ gives a faithful normal trace τ on $L(\mathcal{G}^p)$ extending the integral on $L^\infty(X, \mu)$, with the Hilbert space $L^2(L(\mathcal{G}^p)) = L^2(\mathcal{G}^p)$ giving the standard representation of $(L(\mathcal{G}^p), \tau)$.

Also, note that $L(\mathcal{G}^p)$ is a factor iff \mathcal{G}^p is ergodic, in which case either $L(\mathcal{G}^p) \simeq M_{n \times n}(\mathbb{C})$ (when (X, μ) is the n -points probability space) or $L(\mathcal{G}^p)$ is a II_1 factor (when (X, μ) has no atoms, equivalently when \mathcal{G} has infinitely many elements).

4.2.1. Theorem. *Let $\Gamma \subset \text{Aut}(X, \mu)$, $\Lambda \subset \text{Aut}(Y, \nu)$ be countable groups and $\Delta : (X, \mu) \simeq (Y, \nu)$. The following conditions are equivalent:*

1°. Δ is an orbit equivalence of Γ, Λ , i.e. it takes the Γ -orbits onto the Λ -orbits, a.e.

2°. Δ is a weak equivalence, i.e. it takes the full groups $[\Gamma]$ (or pseudogroup $[\Gamma]^p$) of Γ onto the full group $[\Lambda]$ (resp. pseudogroup $[\Lambda]^p$) of Λ .

3°. The isomorphism implemented by Δ on the function algebras, $\Delta : L^\infty X \simeq L^\infty Y$, extends to an isomorphism of $L([\Gamma]^p)$ onto $L([\Lambda]^p)$.

Proof.

Q.E.D.

Let us end this subsection by mentioning the “full pseudogroup” version of the twisted group measure space construction in 4.1. Thus, let \mathcal{G}^p be a full pseudogroup on (X, μ) and denote \mathcal{U}^p the (commutative) pseudogroup of partial isometries of $L^\infty X$. A \mathcal{U}^p -valued 2-cocycle for \mathcal{G}^p is a map $v : \mathcal{G}^p \times \mathcal{G}^p \rightarrow \mathcal{U}^p$ satisfying the conditions $v_{\phi, \psi} v_{\phi\psi, \rho} = \phi(v_{\psi, \rho}) v_{\phi, \psi\rho}$, $\forall \phi, \psi, \rho \in \mathcal{G}^p$, and $v_{e, e} = 1$. It is easy to see that the axioms together with this normalization condition imply that $v_{e, \phi} = v_{\phi, e} = l(\phi)$, $\forall \phi \in \mathcal{G}^p$. Two such 2-cocycles v, v' are equivalent if there exists $w : \mathcal{G}^p \rightarrow \mathcal{U}^p$ such that $v'_{\phi, \psi} = w_\phi \phi(w_\psi) v_{\phi, \psi} w_{\phi\psi}^*$, $\forall \phi, \psi \in \mathcal{G}^p$.

We associate to the pair (\mathcal{G}^p, v) the von Neumann algebra $L_v(\mathcal{G}^p)$ as follows: Take $A\mathcal{G}^p$ be the vector space of formal finite sums $\sum_\phi a_\phi u_\phi$, with $a_\phi \in A = L^\infty X$ and $\phi \in \mathcal{G}^p$, but with product rule $(a_\phi u_\phi)(a_\psi u_\psi) = a_\phi \phi(a_\psi r(\phi)) u_{\phi\psi}$, then define a trace τ formally the same way as before, etc. It is easy to see that an equivalence of 2-cocycles implements a natural spatial isomorphism between the associated von Neumann algebras. Like with the “un-twisted” version, $L_v(\mathcal{G}^p)$ is a factor iff \mathcal{G} is ergodic.

(I should not forget to do the *orthonormal basis* for $L(\mathcal{G}^p)$. Also, include a remark commenting on the difference between the group measure space construction 4.1 and the full pseudogroup construction 4.2)

4.3. Normalizers and Cartan subalgebras. Let (M, τ) be a finite von Neumann algebra and $A \subset M$ a maximal abelian $*$ -subalgebra of M . The *normalizer* of A in M is the group of unitaries $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$. Let also $\mathcal{N}^p(A) = \{v \in M \mid vv^*, v^*v \in \mathcal{P}(A), vAv^* = Avv^*\}$, where $\mathcal{P}(A)$ denotes the projections (or idempotents) of A . $\mathcal{N}^p(A)$ with its product inherited from M and inverse given by the $*$ -operation is clearly an abstract pseudogroup and we call it the *normalizing pseudogroup*, or the *pseudo-normalizer* of $A \subset M$. Note that $v \in \mathcal{N}^p(A)$ iff there exists $u \in \mathcal{N}(A)$ and $p \in \mathcal{P}(A)$ such that $v = up$.

If $\mathcal{N}_M(A)$ (equivalently $\mathcal{N}^p(A)$) generates M as a von Neumann algebra we say that A is *regular* in M , or that A is a *Cartan subalgebra* of M .

Taking some representation of A as a function algebra, $A = L^\infty X$, we let $\mathcal{G} = \mathcal{G}_{ACM}$ denote the set of all automorphisms of (X, μ) (or of $A = L^\infty X$) of the form $\text{Ad}(u)$ with $u \in \mathcal{N}(A)$, and \mathcal{G}_{ACM}^p be the set of all local isomorphisms of the form $\phi_v = \text{Ad}(v)$, with $v \in \mathcal{N}^p(A)$. Note that $r(\phi_v) = v^*v$, $l(\phi_v) = vv^*$ and $\phi_v\phi_w = \phi_{vw}$, $\forall v, w \in \mathcal{N}^p(A)$. It is easy to see that \mathcal{G}_{ACM} (resp. \mathcal{G}_{ACM}^p) is a full group (resp full pseudogroup), called the *full group* (resp. *full pseudogroup*) of the *Cartan subalgebra* $A \subset M$. Note that $\mathcal{G} = \mathcal{N}/\mathcal{U}$, where $\mathcal{U} = \mathcal{U}(A)$, and similarly $\mathcal{G}^p = \mathcal{N}^p/\mathcal{U}$.

There is also a natural 2-cocycle for \mathcal{G}_{ACM}^p associated with the Cartan inclusion $A \subset M$, as follows: For each $\phi \in \mathcal{G}_{ACM}^p = \mathcal{N}^p(A)/\mathcal{U}^p$ let $v_\phi \in \mathcal{N}^p(A)$ be a representant, with $v_{id_X} = 1$. Define $v = v_{ACM}$ by $v_{\phi, \psi} = v_\phi v_\psi v_{\phi\psi}^*$. It is easy to verify that v is a 2-cocycle for \mathcal{G}^p whose class does not depend on the choice of v_ϕ 's.

Note that if \mathcal{G}^p is an “abstract” full pseudogroup on (X, μ) and v a $\mathcal{U}^p(X)$ -valued 2-cocycle for \mathcal{G}^p , and we denote $A = L(\mathcal{G}_0^p)$, $M = L_v(\mathcal{G}^p)$ then $\mathcal{N}^p(A) = \{au_\phi \mid \phi \in \mathcal{G}^p, a \in \mathcal{U}^p(A)\}$. Thus, the full pseudogroup \mathcal{G}_{ACM}^p associated with the Cartan subalgebra inclusion $L(\mathcal{G}_0^p) \subset L(\mathcal{G}^p)$ can be naturally identified with the initial abstract full pseudogroup \mathcal{G}^p . Also, v_{ACM} clearly coincides (modulo equivalence) with the initial v . Altogether, we have thus shown:

Two Cartan subalgebra inclusions $(A_1 \subset M_1, \tau_1)$, $(A_2 \subset M_2, \tau_2)$ are isomorphic if there exists $\theta : (M_1, \tau_1) \simeq (M_2, \tau_2)$ such that $\theta(A_1) = A_2$. Considering the category of Cartan subalgebra inclusions $A \subset M$, with (M, τ) finite von Neumann algebras and

morphisms given by isomorphisms as above, on the one hand, and the category of pairs (\mathcal{G}^p, v) , consisting of a full pseudogroup and a 2-cocycle on it, with morphisms given by isomorphisms of the full pseudogroups which intertwine the 2-cocycles, we have:

4.3.1. Theorem. *The correspondence $(A \subset M) \rightarrow (\mathcal{G}_{A \subset M}^p, v_{A \subset M})$ gives an equivalence of categories, whose inverse is $(\mathcal{G}^p, v) \rightarrow (L(\mathcal{G}_0^p) \subset L_v(\mathcal{G}^p))$.*

4.3.2. Lemma. *Let (M, τ) be a finite von Neumann algebra and $A \subset M$ a maximal abelian $*$ -subalgebra. Let $B \subset M$ be a von Neumann subalgebra containing A . If $v \in \mathcal{N}_M^p(A)$ then there exists a projection $q \in A, q \leq v^*v$ such that $E_B(v) = vq$. Thus, $E_B(\mathcal{N}_M^p(A)) = \mathcal{N}_B^p(A)$.*

4.3.3. Corollary. *With $A \subset B \subset M$ as in 4.3.2, let M_0 (resp B_0) denote the von Neumann algebra generated by $\mathcal{N}_M^p(A)$ (resp $\mathcal{N}_B^p(A)$). Then $E_B E_{M_0} = E_{M_0} E_B = E_{B_0}$. In particular, if A is Cartan in M then it is Cartan in B .*

4.4. Amplification of a Cartan subalgebra inclusion. If M is a II_1 factor and $t > 0$ then for any $n \geq m \geq t$ and any projections $p \in M_{n \times n}(M), q \in M_{m \times m}(M)$ of (normalized) trace $\tau(p) = t/n, \tau(q) = t/m$, one has $pM_{n \times n}(M)p \simeq qM_{m \times m}(M)q$. Indeed, because if we regard $M_{m \times m}(M)$ as a ‘‘corner’’ of $M_{n \times n}(M)$ then p, q have the same trace in $M_{n \times n}(M)$, so they are conjugate by a unitary U in $M_{n \times n}(M)$, which implements an isomorphism between $pM_{n \times n}(M)p$ and $qM_{m \times m}(M)q$. One denotes by M^t this common (up to isomorphism) II_1 factor and one calls it the *amplification* of M by t .

Similarly, if $A \subset M$ is a Cartan subalgebra of the II_1 factor M then $(A \subset M)^t = (A^t \subset M^t)$ denotes the (isomorphism class of the) Cartan subalgebra inclusion $p(A \otimes D_n \subset M \otimes M_{n \times n}(\mathbb{C}))p$ where $n \geq t, D_n$ is the diagonal subalgebra of $M_{n \times n}(\mathbb{C})$ and $p \in A \otimes D_n$ is a projection of trace $\tau(p) = t/n$. In this case, the fact that the isomorphism class of $(A \subset M)^t$ doesn’t depend on the choice of n, p follows from a lemma of H. Dye ([D63]), showing that if M_0 is a II_1 factor and $A_0 \subset M_0$ is a Cartan subalgebra, then two projections $p, q \in A_0$ having the same trace are conjugate by a unitary element in the normalizer of A_0 in M_0 .

$(A \subset M)^t$ is called the *t-amplification* of $A \subset M$. We clearly have $\mathcal{G}_{(A \subset M)^t}^p = (\mathcal{G}_{A \subset M}^p)^t, \mathcal{R}_{(A \subset M)^t} = \mathcal{R}_{A \subset M}^t$ and if $\mathcal{G}^p, \mathcal{R}$ correspond with one another then so do $\mathcal{R}^t, (\mathcal{G}^p)^t, \forall t$. Note that $((A \subset M)^t)^s = (A \subset M)^{st}, (\mathcal{G}^t)^s = \mathcal{G}^{ts}, (\mathcal{R}^t)^s = \mathcal{R}^{ts}, \forall t, s > 0$.

4.5. Basic construction for Cartan subalgebra inclusions. Let (M, τ) be a finite von Neumann algebra and $A \subset M$ a Cartan subalgebra. Denote by e_A the orthogonal projection of L^2M onto L^2A . More generally, if $v \in \mathcal{N}^p(A)$ then denote by e_{vA} the orthogonal projection of L^2M onto vL^2A . Thus, $e_{vA} = ve_A v^*$.

4.5.1. Proposition. *Let $\{v_n\}_n \subset \mathcal{N}^p(A)$ be an orthonormal basis over A . Then the abelian von Neumann subalgebra $\tilde{A} = \{\sum_n a_n e_{v_n A} \mid a_n \in Av_n v_n^*, \sup_n \|a_n\| < \infty\}$ of $\mathcal{B}(L^2 M)$ is maximal abelian and it coincides with the von Neumann algebra generated by A and $J_M A J_M$. Also, \tilde{A} is the smallest von Neumann algebra which contains A , e_A and is normalized by $\text{Ad}(v)$, $\forall v \in \mathcal{N}(A)$.*

If \mathcal{G}^p is a full pseudogroup on (X, μ) we denote $(\tilde{X}, \tilde{\mu})$ the measurable space...

5. AMENABLE ACTIONS AND THEIR CLASSIFICATION UP TO OE

5.1. *Definition.* A Cartan subalgebra inclusion $A \subset M$ is *approximately finite dimensional* (AFD) if for any finite set $F \subset M$ and any $\varepsilon > 0$ there exist matrix units $\{e_{ij}^k \mid 1 \leq i, j \leq n_k, 1 \leq k \leq m\} \subset \mathcal{N}^p(A)$, $e_{ii}^k \in A$, such that if B denotes the finite dimensional von Neumann algebra generated by e_{ij}^k then $\|E_B(x) - x\|_2 \leq \varepsilon$, $\forall x \in F$.

A full pseudogroup \mathcal{G}^p (resp. full group \mathcal{G}) is AFD if the corresponding Cartan subalgebra inclusion $L(\mathcal{G}_0) \subset L(\mathcal{G}^p)$ is AFD. Note that this amounts to requiring that for all $F \subset \mathcal{G}^p$ and $\varepsilon > 0$ there exists a finite sub-pseudogroup $\mathcal{F} \subset \mathcal{G}^p$ such that $\forall \phi \in F$, $\exists \psi \in \mathcal{F}$ with $\mu(r(\psi) \setminus i(\phi^{-1}\psi)), \mu(r(\phi) \setminus i(\psi^{-1}\phi)) \leq \varepsilon$. It is a trivial exercise to show that this condition on a full pseudogroup \mathcal{G}^p is equivalent to \mathcal{G} having the property that for all $F \subset \mathcal{G}$ finite and $\varepsilon > 0$ there exists a finite subgroup $\mathcal{F} \subset \mathcal{G}$ such that $\forall \phi \in F$, $\exists \psi \in \mathcal{F}$ with $\mu(i(\phi^{-1}\psi)) \geq 1 - \varepsilon$.

5.2. *Definition.* A Cartan subalgebra $A \subset M$ is *amenable* if there exists an $\text{Ad}\mathcal{N}(A)$ invariant state ϕ (called an *invariant mean*) on \tilde{A} . A full group \mathcal{G} (resp full pseudogroup \mathcal{G}^p) is *amenable* if $L(\mathcal{G}_0^p) \subset L(\mathcal{G}^p)$ is amenable. Note that this amounts to \tilde{A} having a \mathcal{G} -invariant mean. An action $\sigma : \Gamma \rightarrow \text{Aut}(X, \mu)$ is amenable if $[\sigma(\Gamma)]$ is amenable.

5.3. Proposition. *Let $A \subset M$ be a Cartan subalgebra inclusion.*

1°. *Let $\Gamma \subset \mathcal{N}(A)$ be a subgroup such that $(A \cup \Gamma)'' = M$. If Γ is amenable then $A \subset M$ is amenable. Conversely, if Γ acts freely and $A \subset M$ is amenable then Γ is amenable.*

2°. *$A \subset M$ is amenable iff M is amenable as a vN algebra.*

3°. *If $A \subset M$ is amenable and $p \in \mathcal{P}(A)$ then $Ap \subset pMp$ is amenable.*

5.4. Theorem. 1°. *$A \subset M$ (or \mathcal{G}^p) amenable iff AFD.*

2°. *If M is a factor and $A \subset M$ amenable, then either $(A \subset M) = (D \subset M_{n \times n}(\mathbb{C}))$, or $(A \subset M) \simeq (D \subset R)$.*

Proof. It is an easy exercise to deduce part 2° from part 1°, so we will only prove the latter.

Step 1 (Day's trick). We first prove that given any $v_1, v_2, \dots, v_n \in \mathcal{N}(A)$ and any $\varepsilon > 0$ there exists $b \in \tilde{A}_+$ such that $\tilde{\tau}(b) = 1$ and $\|bv_i v_i^* - b\|_{1, \tilde{\tau}} < \varepsilon$. To do this, we use an argument similar to the one used in the proof of (iv) \Rightarrow (iii) in 1.5.1. Let \mathcal{V} be the set on n -tuples $(\psi - \psi(u_i^* \cdot u_i))_i$ with $\psi \in \tilde{A}_*$ a normal state on \tilde{A} . Then \mathcal{V} is a (bounded) convex subset of $(\tilde{A}_*)^n \subset (\tilde{A}^*)^n = (\tilde{A}^n)^*$. We claim that $0 = (0, \dots, 0)$ is in the (norm) closure of \mathcal{V} in the Banach space $(\tilde{A}_*)^n$. Indeed, for if not then by the Hahn-Banach theorem there exists $(x_1, \dots, x_n) \in ((\tilde{A}_*)^n)^* = ((\tilde{A}^*)^n)^* = \tilde{A}^n$ such that

$$\text{Re} \sum_{i=1}^n (\psi(x_i) - \psi(u_i^* x_i u_i)) \geq \alpha > 0, \forall \psi \in \mathcal{L}.$$

But this would then hold true equally well for all weak limits of normal states $\psi \in \tilde{A}_*$ in \tilde{A}^* , thus for φ . In particular, it holds true for any $\text{Ad}\mathcal{N}(A)$ -invariant mean φ on \tilde{A} , for which we thus get

$$0 = \text{Re}\sum_{i=1}^n (\phi(x_i) - \phi(u_i^* x_i u_i)) \geq \alpha > 0,$$

a contradiction.

Thus, there must exist a state $\psi \in \tilde{A}_*$ such that $\sum_i \|\psi - \psi(u_i^* \cdot u_i)\| < \varepsilon$. Since the states of the form $\tilde{\tau}(b \cdot)$ with $b \in \tilde{A}_+$, $\tilde{\tau}(b) = 1$ are dense in the set of normal states on \tilde{A} , it follows that there exists $b \in \tilde{A}_+$ satisfying $\tilde{\tau}(b) = 1$ and

$$(5.4.1) \quad \sum_i \|\tilde{\tau}(b \cdot) - \tilde{\tau}(bu_i^* \cdot u_i)\| < \varepsilon,$$

the norm being taken in the Banach space $\tilde{A}_* \subset \tilde{A}^*$. But $\|\tilde{\tau}(b \cdot) - \tilde{\tau}(bu_i^* \cdot u_i)\| = \|b - u_i b u_i^*\|_{1, \tilde{\tau}}$, so that (5.4.1) implies

$$(5.4.1') \quad \sum_i \|b - u_i b u_i^*\|_{1, \tilde{\tau}} < \varepsilon.$$

Step 2 (Namioka's trick). We now show that for any finite set $F \subset \mathcal{N}(A)$ and any $\varepsilon > 0$ there exists a projection $e \in \tilde{A}$ such that $\tilde{\tau}(e) < \infty$ and $\sum_i \|e - u_i e u_i^*\|_{2, \tilde{\tau}}^2 < \varepsilon \|e\|_{2, \tilde{\tau}}^2$. Indeed, with $b \in \tilde{A}_+$ as in *Step 1*, by Lemma 1.5.2 it follows that there exists $s > 0$ such that if we denote by $e = e_s(b)$ the spectral projection (or if one prefers the level set) of b corresponding to the interval $[s, \infty)$ then

$$(5.4.2) \quad \sum_i \|e - u_i e u_i^*\|_{2, \tilde{\tau}}^2 < \varepsilon \|e\|_{2, \tilde{\tau}}^2.$$

Note that $\tilde{\tau}(e) < \infty$.

Step 3 (Local AFD approximation). We now prove that given any finite set $\{v_i\}_i \subset \mathcal{N}^p(A)$ and any $\varepsilon > 0$ there exists matrix units $\{e_{kl}\}_{k,l} \subset \mathcal{N}^p(A)$, $e_{kk} \in A$, such that if we denote $s_0 = \sum_k e_{kk}$ and N_0 the algebra generated by As_0 and $\{e_{kl}\}_{k,l}$ then $s_0 v_i s_0 \in N_0$ and $\sum_i \|[s_0, v_i]\|_2^2 < \varepsilon \|s_0\|_2^2$. It is clearly sufficient to prove this for $v_i = u_i$ unitary elements, thus from $\mathcal{N}(A)$. Let then e be the finite projection in \tilde{A} satisfying $\sum_i \|e - u_i e u_i^*\|_{2, \tilde{\tau}}^2 < \varepsilon \|e\|_{2, \tilde{\tau}}^2$, obtained in *Step 2*.

Since any projection in \tilde{A} is of the form $\sum_n e_{v_n A}$ for some orthonormal system $\{v_n\}_n \subset \mathcal{N}^p(A)$ and since $\tilde{\tau}(e) < \infty$, we may assume in addition that $e = \sum_{n=1}^m e_{v_n A}$ (finite sum). Let $\{q_j\}_{j \geq 1} \subset A$ be a partition of 1 such that $q_j(v_k^* u_i v_l) q_j$ is either equal to 0 or to a unitary element in $A q_j$, for all i, k, l and for all $j \geq 1$. By (5.4.2) and Pythagora's Theorem, it follows that

$$\sum_j (\sum_i \|(e - u_i e u_i^*) J q_j J\|_{2, \tilde{\tau}}^2 < \varepsilon \sum_j \|e J q_j J\|_{2, \tilde{\tau}}^2.$$

Thus, there must exist $j \geq 1$ such that $q = q_j$ satisfies

$$(5.4.3) \quad \Sigma_i \|(e - u_i e u_i^*) J q J\|_{2, \tilde{\tau}}^2 < \varepsilon \|e J q J\|_{2, \tilde{\tau}}^2.$$

But $e J q J = \Sigma_n e_{v_n A} J q J = \Sigma_n v_n e_A v_n^* J q J = \Sigma_n v_n q e_A v_n^*$. By our choice of q , for each n we have either $v_n q = 0$ or $q(v_n^* v_n)q = q$. Thus, after a suitable relabeling, we may assume e in (5.4.3) is of the form $e = \Sigma_n v_n e_A v_n^*$ with $v_k^* v_l = \delta_{lk} q$ and $v_k^* u_i v_l \in A q$, $\forall i, l, k$. Thus, for this new e we have

$$(5.4.3') \quad \Sigma_i \|e - u_i e u_i^*\|_{2, \tilde{\tau}}^2 < \varepsilon \|e\|_{2, \tilde{\tau}}^2.$$

Denote $e_{kl}^0 = v_k v_l^*$, $s_0 = \Sigma_k e_{kk}^0$ and N_0 the algebra generated by the matrix units $\{e_{kl}^0\}_{kl}$ and $A s_0$. Then $s_0 u_i s_0 \in N_0$. We claim that the left hand term of (5.4.2') is equal to $\Sigma_i \|s_0 - u_i s_0 u_i^*\|_2^2$ while the right hand term is equal to $\|s_0\|_2^2$, so that altogether (5.4.3') amounts to

$$(5.4.3'') \quad \Sigma_i \|s_0 - u_i s_0 u_i^*\|_2^2 < \varepsilon \|s_0\|_2^2.$$

Indeed, by the definition of $\tilde{\tau}$ we have

$$\|e\|_{2, \tilde{\tau}}^2 = \tilde{\tau}(\Sigma_k v_k e v_k^*) = \Sigma_k \tau(v_k v_k^*) = \tau(s_0)$$

and similarly

$$\begin{aligned} \|e - u_i e u_i^*\|_{2, \tilde{\tau}}^2 &= 2 \Sigma_k \tilde{\tau}(v_k e_A v_k^*) - 2 \Sigma_{k,l} \tilde{\tau}(v_k e_A v_k^* u_i v_l e_A v_l^*) \\ &= 2 \Sigma_k (v_k v_k^*) - 2 \Sigma_{k,l} \tau(v_k v_k^* u_i v_l v_l^* u_i^*) = \|s_0 - u_i s_0 u_i^*\|_2^2 \end{aligned}$$

Step 4 (Maximality argument). Consider now the set \mathcal{A} of all families of subalgebras $\{N_j\}_j$ of M with mutually orthogonal units $s_j = 1_{N_j}$, with each N_j generated by $A s_j$ and by a finite set of matrix units $\{e_{kl}^j\}_{k,l} \subset \mathcal{N}^p(A)$, such that $s_j u_i s_j \in N_j, \forall i, j$ and

$$(5.4.4) \quad \Sigma_i \|\Sigma_j s_j u_i s_j - (u_i - (1 - s) u_i (1 - s))\|_2^2 \leq \varepsilon \|s\|_2^2,$$

where $s = \Sigma_j s_j$. The set \mathcal{A} is clearly inductively ordered with respect to inclusion. Let $\{N_j\}_j$ be a maximal family and suppose the units s_j do not fill up the unity of M . Take $p = 1 - s \in A$. Then $A p \subset p M p$ is still amenable by Proposition 5.3, so we can apply Step 3 above to the finite set $v_i = p u_i p$, to get a non-zero projection $s_0 \subset A p$ and a set of matrix units $\{e_{kl}^0\}_{k,l} \subset \mathcal{N}^p(A)$ with $s_0 = \Sigma_k e_{kk}^0$ such that $s_0 u_i s_0 \in N_0 = \Sigma_{k,l} A s_0$, $\forall i$, and $\Sigma_i \|[p u_i p, s_0]\|_2^2 \leq \varepsilon \|s_0\|_2^2$. Together with (5.4.4), by using Pythagora this gives

$$\Sigma_i \|(s_0 u_i s_0 + \Sigma_j s_j u_i s_j) - (u_i - (1 - (s + s_0)) u_i (1 - (s + s_0)))\|_2^2$$

$$\begin{aligned}
&= \sum_i \|\sum_j s_j u_i s_j - (u_i - (1-s)u_i(1-s))\|_2^2 + \sum_i \|s_0 u_i s_0 - (p u_i p - (p-s_0)u_i(p-s_0))\|_2^2 \\
&= \sum_i \|\sum_j s_j u_i s_j - (u_i - (1-s)u_i(1-s))\|_2^2 + \sum_i \|[p u_i p, s_0]\|_2^2 \\
&\leq \varepsilon (\|s\|_2^2 + \|s_0\|_2^2) = \varepsilon \|s + s_0\|_2^2.
\end{aligned}$$

But this contradicts the maximality of $\{N_j\}_j$. Thus, we must have $\sum_j s_j = 1$.

This clearly finishes the proof, since we can now take any sufficiently large finite subset $\{N_j\}_{1 \leq j \leq n}$ of the maximal family and the algebra $N = \sum_{j=1}^m N_j$ will be finite dimensional over A , will be generated by a finite set of matrix units from the pseudonormalizer $\mathcal{N}^p(A)$ and will still approximate the given finite set u_i . Q.E.D.

6. THE FIRST COHOMOLOGY GROUP OF AN ACTION

6.1. 1-cohomology for groups of automorphisms. Let $\sigma : \Gamma \rightarrow \text{Aut}(X, \mu)$ be a measure preserving action of a discrete group Γ on the standard probability space (X, μ) and denote $A = L^\infty X$, $\mathcal{U}(A) = \{u \in A \mid uu^* = 1\}$. A function $w : \Gamma \rightarrow \mathcal{U}(A)$ satisfying $w_g \sigma_g(w_h) = w_{gh}, \forall g, h \in \Gamma$, is called a 1-cocycle for σ . Note that a scalar valued function $w : \Gamma \rightarrow \mathcal{U}(A)$ is a 1-cocycle iff $w \in \text{Char}(\Gamma)$.

Two 1-cocycles w, w' are *cohomologous*, $w \sim w'$, if there exists $u \in \mathcal{U}(A)$ such that $w'_g = u^* w_g \sigma_g(u), \forall g \in G$. A 1-cocycle w is *coboundary* if $w \sim \mathbf{1}$, where $\mathbf{1}_g = 1, \forall g$.

Denote by $Z^1(\sigma)$ the set of 1-cocycles for σ , endowed with the structure of a topological (commutative) group given by point multiplication and pointwise convergence in norm $\|\cdot\|_2$. Denote by $B^1(\sigma) \subset Z^1(\sigma)$ the subgroup of coboundaries and by $H^1(\sigma)$ the quotient group $Z^1(\sigma)/B^1(\sigma) = Z^1(\sigma)/\sim$, called the 1'st cohomology group of σ . Note that $\text{Char}(\Gamma)$ with its usual topology is canonically embedded as a compact subgroup of $Z^1(\sigma)$, via the map $\gamma \mapsto w^\gamma$, where $w_g^\gamma = \gamma(g)1, g \in \Gamma$. Its image in $H^1(\sigma)$ is a compact subgroup. If in addition σ is weakly mixing, then this image is actually faithful:

6.1.1. Lemma. *If σ is weakly mixing then the group morphism $\gamma \mapsto w^\gamma$ is 1 to 1 and continuous from $\text{Char}(\Gamma)$ into $H^1(\sigma)$.*

Proof. If $w_1(g) = u^* w_2(g) \sigma_g(u), \forall g \in \Gamma$ then $\sigma_g(u) \in \mathbb{C}u, \forall g \in \Gamma$ and since σ is weakly mixing, this implies $u \in \mathbb{C}1$ so $w_1 = w_2$.

Q.E.D.

Let us note that under appropriate mixing conditions the fact that a cocycle w is scalar on a subgroup $H \subset \Gamma$ automatically entails that w_g for all elements $g \in \Gamma$ which "almost" normalize H :

6.1.2. Lemma. *Let $H \subset \Gamma$ be an infinite subgroup of Γ and $w \in Z^1(\sigma)$ be so that $w|_H \in \text{Char}(H)$. If $g \in \Gamma$ is such that $H' = g^{-1}Hg \cap H$ is infinite and σ is weak mixing on H' then $w_g \in \mathbb{C}1$.*

Proof. Take $k \in H'$ and put $h = gkg^{-1} \in H$. Then $hg = gk$. The 1-cocycle relation yields $w_h \sigma_h(w_g) = w_g \sigma_g(w_k)$. Since $w_h, w_k \in \mathbb{C}1$, this implies $\sigma_h(w_g) \in \mathbb{C}w_g$. Thus, $\sigma_h(w_g) \in \mathbb{C}w_g, \forall h \in gH'g^{-1}$. Since $\sigma|_{gH'g^{-1}}$ is weakly mixing (because $\sigma|_{H'}$ is weakly mixing) this implies $w_g \in \mathbb{C}1$.

Q.E.D.

The above lemma justifies considering the following:

6.1.3. Definition. Let $H \subset \Gamma$ be an inclusion of infinite groups. The *w-normalizer* of H in Γ is the group... The *wq-normalizer* of H in Γ is...

6.2. Automorphisms associated with 1-cocycles. The groups $B^1(\sigma), Z^1(\sigma), H^1(\sigma)$ were first considered in I.M. Singer, who also noticed that they can be identified with certain groups of automorphisms of the finite von Neumann algebra $M = L^\infty X \rtimes_\sigma \Gamma$, as follows.

Let $M = L^\infty X \rtimes \Gamma$ and $A = L^\infty X \subset M$. Denote by $\text{Aut}_0(M; A)$ the group of automorphisms of M that leave all elements of A fixed, endowed with the topology of pointwise convergence in norm $\|\cdot\|_2$ (the topology it inherits from $\text{Aut}(M, \tau)$). If $\theta \in \text{Aut}_0(M; A)$ then $w_g^\theta = \theta(u_g)u_g^*$, $g \in \Gamma$, is a 1-cocycle, where $\{u_g\}_g \subset M$ denote the canonical unitaries implementing the action σ . Conversely, if $w \in Z^1(\sigma)$ then $\theta^w(au_g) = aw_gu_g$, $a \in A, g \in \Gamma$, defines an automorphism of M that fixes A . Clearly $\theta \mapsto w^\theta$, $w \mapsto \theta^w$ are group morphisms and are inverse one another, thus identifying $Z^1(\sigma)$ with $\text{Aut}_0(M; A)$ as topological groups, with $B^1(\sigma)$ corresponding to the inner automorphism group $\text{Int}_0(M; A) = \{\text{Ad}(u) \mid u \in \mathcal{U}(A)\}$. Thus, $H^1(\sigma)$ is naturally isomorphic to $\text{Out}_0(M; A) \stackrel{\text{def}}{=} \text{Aut}_0(M; A)/\text{Int}_0(M; A)$.

The groups $\text{Aut}_0(M; A)$, $\text{Int}_0(M; A)$, $\text{Out}_0(M; A)$ make actually sense for any inclusion $A \subset M$ consisting of a II_1 factor M with a *Cartan subalgebra* A .

6.3. 1-cohomology for full pseudogroups. Let \mathcal{G} be a full pseudogroup acting on the probability space (X, μ) and denote $A = L^\infty(X, \mu)$, as before. A 1-cocycle for \mathcal{G} is a map $w : \mathcal{G} \rightarrow_p \mathcal{U}(A)$ satisfying the relation $w_\phi \phi(w_\psi) = w_{\phi\psi}$, $\forall \phi, \psi \in \mathcal{G}$. In particular, this implies that the support of w_ϕ , $w_\phi w_\phi^*$, is equal to the range $r(\phi)$ of ϕ . Thus, $w_{id_Y} = \chi_Y$, $\forall Y \subset X$ measurable.

We denote by $Z^1(\mathcal{G})$ the set of all 1-cocycles and endow it with the (commutative) semigroup structure given by point multiplication. We denote by $\mathbf{1}$ the 1-cocycle given by $\mathbf{1}_\phi = r(\phi)$, $\forall \phi \in \mathcal{G}$. If we let $(w^{-1})_\phi = w_\phi^*$ then we clearly have $ww^{-1} = \mathbf{1}$ and $\mathbf{1}w = w$, $\forall w \in Z^1(\mathcal{G})$. Thus, together also with the topology given by pointwise norm $\|\cdot\|_2$ -convergence, $Z^1(\mathcal{G})$ is a commutative Polish group.

Two 1-cocycles w_1, w_2 are *cohomologous*, $w_1 \sim w_2$, if there exists $u \in \mathcal{U}(A)$ such that $w_2(\phi) = u^*w_1(\phi)\phi(u)$, $\forall \phi \in \mathcal{G}$. A 1-cocycle w cohomologous to $\mathbf{1}$ is called a *coboundary* for \mathcal{G} and the set of coboundaries is denoted $B^1(\mathcal{G})$. It is clearly a subgroup of $Z^1(\mathcal{G})$. We denote the quotient group $H^1(\mathcal{G}) \stackrel{\text{def}}{=} Z^1(\mathcal{G})/B^1(\mathcal{G}) = Z^1(\mathcal{G})/\sim_c$ and call it the 1'st *cohomology group* of \mathcal{G} .

By the correspondence between countably generated full pseudogroups and countable m.p. standard equivalence relations described in Section 6.2, one can alternatively view the 1-cohomology groups $Z^1(\mathcal{G}), B^1(\mathcal{G}), H^1(\mathcal{G})$ as associated to the equivalence relation $\mathcal{R} = \mathcal{R}_\mathcal{G}$.

Let now $A \subset M$ be a II_1 factor with a Cartan subalgebra. If $\theta \in \text{Aut}_0(M; A)$ and $\phi_v = \text{Ad}(v) \in \mathcal{G}_{ACM}$ for some $v \in \mathcal{GN}_M(A)$ then $w^\theta(\phi_v) = \theta(v)v^*$ is a well defined 1-cocycle for \mathcal{G} . Conversely, if $w \in H^1(\mathcal{G})$ then there exists a unique automorphism $\theta^w \in \text{Aut}_0(M; A)$ satisfying $\theta^w(av) = aw_{\phi_v}v$, $\forall a \in A, v \in \mathcal{GN}_M(A)$.

6.3.1. Proposition. $\theta \mapsto w^\theta$ is an isomorphism of topological groups, from $\text{Aut}_0(M; A)$ onto $Z^1(\mathcal{G}_{ACM})$, that takes $\text{Int}_0(M; A) = \{\text{Ad}(u) \mid u \in \mathcal{U}(A)\}$ onto $B^1(\mathcal{G}_{ACM})$ and whose inverse is $w \mapsto \theta^w$. Thus, $\theta \mapsto w^\theta$ implements an isomorphism between the topological groups $\text{Out}_0(M; A) = \text{Aut}_0(M; A)/\text{Int}_0(M; A)$ and $H^1(\mathcal{G}_{ACM})$. ■

Proof. This is trivial by the definitions.

Q.E.D.

It is an easy exercise to show that if $\theta \in \text{Aut}_0(M; A)$ satisfies $\theta|_{pMp} = \text{Ad}(u)|_{pMp}$ for some $p \in \mathcal{P}(A)$, $u \in \mathcal{U}(A)$ then $\theta \in \text{Int}_0(M; A)$. Thus, $\theta \mapsto \theta|_{pMp}$ defines an isomorphism from $\text{Out}_0(M; A)$ onto $\text{Out}_0(pMp; Ap)$. Applying this to the Cartan subalgebra inclusion $L(\mathcal{G}_0) \subset L(\mathcal{G})$ for \mathcal{G} an ergodic full pseudogroup acting on the non-atomic probability space, from 1.5.1 we get: $H^1(\mathcal{G})$ is naturally isomorphic to $H^1(\mathcal{G}^t), \forall t > 0$. In particular, since 1.5.1 also implies $H^1(\sigma) = H^1(\mathcal{G}_\sigma)$, it follows that $H^1(\sigma)$ is invariant to stable orbit equivalence. We have thus shown:

6.3.2. Corollary. 1°. $H^1(\mathcal{G}^t)$ is naturally isomorphic to $H^1(\mathcal{G}), \forall t > 0$.

2°. If σ is a free ergodic measure preserving action then $H^1(\sigma) = H^1(\mathcal{G}_\sigma)$ and $H^1(\sigma)$ is invariant to stable orbit equivalence. Also, $Z^1(\sigma) = Z^1(\mathcal{G}_\sigma)$ and $Z^1(\sigma)$ is invariant to orbit equivalence.

6.4. The closure of $B^1(\mathcal{G})$ in $Z^1(\mathcal{G})$. Given any ergodic full pseudogroup \mathcal{G} , the groups $B^1(\mathcal{G}) \simeq \text{Int}_0(M; A)$ are naturally isomorphic to $\mathcal{U}(A)/\mathbb{T}$, where $A = L(\mathcal{G}_0), M = L(\mathcal{G})$. But this isomorphism doesn't always carry the topology that $B^1(\mathcal{G})$ (resp. $\text{Int}_0(M; A)$) inherits from $Z^1(\mathcal{G})$ (resp. $\text{Aut}_0(M; A)$) onto the quotient of the $\|\cdot\|_2$ -topology on $\mathcal{U}(A)/\mathbb{T}$.

6.4.1. Proposition. Let $A \subset M$ be a II_1 factor with a Cartan subalgebra. The following conditions are equivalent:

(a). $H^1(\mathcal{G}_{A \subset M})$ is a Polish group (equivalently $H^1(\mathcal{G}_{A \subset M})$ is separate), i.e. $B^1(\mathcal{G}_{A \subset M})$ is closed in $Z^1(\mathcal{G}_{A \subset M})$. ■

(b). $\text{Int}_0(M; A)$ is closed in $\text{Aut}_0(M; A)$.

(c). The action of $\mathcal{G}_{A \subset M}$ on A is strongly ergodic, i.e. it has no non-trivial asymptotically invariant sequences.

Moreover, if $M = A \rtimes_\sigma \Gamma$ for some free action σ of a group Γ on (A, τ) , then the above conditions are equivalent to σ being strongly ergodic.

Proof. (a) \Leftrightarrow (b) follows from 6.3.1. Then notice that (b) \Leftrightarrow (d) is a relative version of Connes' result in ([C75]), showing that “ $\text{Int}(N)$ is closed in $\text{Aut}(N)$ iff N has no non-trivial central sequences” for II_1 factors N . Thus, a proof of (b) \Leftrightarrow (d) is obtained by following the argument in ([C75]), but replacing everywhere $\text{Int}(N)$ by $\text{Int}_0(M; A)$, $\text{Aut}(N)$ by $\text{Aut}_0(M; A)$ and “non-trivial central sequences of N ” by “non-trivial central sequences of M that are contained in A ”.

To prove the last part, note that σ strongly ergodic iff $\{u_g\}'_g \cap A^\omega = \mathbb{C}$, where $\{u_g\}_g \subset M$ denote the canonical unitaries implementing the action σ of G on A . But $\{u_g\}'_g \cap A^\omega = (A \cup \{u_g\}_g)' \cap A^\omega = M' \cap A^\omega$, hence strong ergodicity of σ is equivalent to (d). Q.E.D.

6.4.2. Proposition. *Assume Γ has an infinite subgroup $H \subset \Gamma$ such that the pair (Γ, H) has the relative property (T). If σ is a free m.p. action of Γ on the probability space such that $\sigma|_H$ is ergodic, then σ is strongly ergodic, equivalently $B^1(\sigma)$ is closed in $Z^1(\sigma)$. Moreover, the subgroup $Z_H^1(\sigma) \stackrel{\text{def}}{=} \{w \in Z^1(\sigma) \mid w|_H \sim \mathbf{1}_H\}$ is open and closed in $Z^1(\sigma)$.*

Proof. Since (Γ, H) has the relative property (T), by ([Jol02]) there exist a finite subset $F \subset G$ and $\delta > 0$ such that if $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$, $\xi \in \mathcal{H}$, $\|\xi\|_2 = 1$ satisfy $\|\pi_g(\xi) - \xi\|_2 \leq \delta, \forall g \in F$ then $\|\pi_h(\xi) - \xi\|_2 \leq 1/2, \forall h \in H$ and $\pi|_H$ has a non-trivial fixed vector.

If σ is not strongly ergodic then there exists $p \in \mathcal{P}(A)$ such that $\tau(p) = 1/2$ and $\|\sigma_g(p) - p\|_2 \leq \delta/2, \forall g \in F$. But then $u = 1 - 2p$ satisfies $\tau(u) = 0$ and $\|\sigma_g(u) - u\|_2 \leq \delta, \forall g \in F$. Taking π to be the G -representation induced by σ on $L^2(A, \tau) \ominus \mathbb{C}1$, it follows that $L^2(A, \tau) \ominus \mathbb{C}1$ contains a non-trivial vector fixed by $\sigma|_H$. But this contradicts the ergodicity of $\sigma|_H$.

Let now $M = A \rtimes_{\sigma} \Gamma$ and $\theta = \theta^w \in \text{Aut}_0(M; A)$ be the automorphism associated to some $w \in Z^1(\sigma)$ satisfying $\|\theta(u_g) - u_g\|_2 = \|w_g - 1\|_2 \leq \delta, \forall g \in F$. Then the unitary representation $\pi : G \rightarrow \mathcal{U}(L^2(M, \tau))$ defined by $\pi_g(\xi) = u_g \xi \theta(u_g^*)$ satisfies $\|\pi_g(\hat{1}) - \hat{1}\|_2 = \|w_g - 1\|_2 \leq \delta, \forall g \in F$. Thus, $\|w_h - 1\|_2 = \|\pi_h(\hat{1}) - \hat{1}\|_2 \leq 1/2$ implying

$$\|\theta(vu_h) - vu_h\|_2 = \|\theta(u_h) - u_h\|_2 \leq 1/2, \forall h \in H, v \in \mathcal{U}(A).$$

It follows that if b denotes the element of minimal norm $\|\cdot\|_2$ in $\overline{\text{co}}^w \{u_h^* v^* \theta(vu_h) \mid h \in H, v \in \mathcal{U}(A)\}$ then $\|b - 1\|_2 \leq 1/2$ and $vu_h b = b\theta(vu_h) = bw_h vu_h, \forall h \in H, v \in \mathcal{U}(A)$. But this implies $b \neq 0$ and $xb = b\theta(x), \forall x \in N = A \rtimes_{\sigma|_H} H$. In particular $[b, A] = 0$ so $b \in A \subset N$. Since N is a factor (because $\sigma|_H$ is ergodic), this implies b is a scalar multiple of a unitary element u in A satisfying $w_h = u^* \sigma_h(u), \forall h \in H$. Thus $w \in Z_H^1(\sigma)$, showing that $Z_H^1(\sigma)$ is open (thus closed too). Q.E.D.

6.4.3. Corollary. *Assume Γ has an infinite rigid subgroup $H \subset \Gamma$. Let $\Gamma \curvearrowright^{\sigma} (X, \mu)$ be a m.p. action with $\sigma|_H$ ergodic. If $w, w' \in Z^1(\sigma)$ are in the same connected component of $Z^1(\sigma)$ then $w|_H \sim w'|_H$.*

Proof. If w, w' are in the same connected component of $Z^1(\sigma)$ then $w'w^{-1}$ is in the connected component \mathcal{C} of 1 in $Z^1(\sigma)$. But then $Z_H^1(\sigma)$ is open (by 6.4.2), contains 1 and its complement is also open, implying that $\mathcal{C} \subset Z_H^1(\sigma)$, i.e. $w'w^{-1} \sim 1$, equivalently $w \sim w'$. Q.E.D.

6.5. Calculation of H^1 for malleable actions. Throughout this section let $\Gamma \curvearrowright^{\sigma} (X, \mu)$ denote a m.p. action of the discrete group Γ on the probability space (X, μ) .

6.5.1. Definition. σ is malleable if the flip automorphism α_1 on $X \times X$, defined by $\alpha_1(t, t') = (t', t), t, t' \in X$, is in the connected component of the identity id_X in

the Polish group $\tilde{\sigma}(\Gamma)' \cap \text{Aut}(X \times X, \mu \times \mu)$, where $\tilde{\sigma}$ is the diagonal product action $\tilde{\sigma}g = \sigma_g \times \sigma_g, g \in \Gamma$.

6.5.2. Theorem. *Let Γ be a countable discrete group with an infinite subgroup $H \subset \Gamma$ such that (Γ, H) has the relative property (T). Let σ be a free ergodic m.p. action of Γ on the probability space. Assume that σ is malleable w -mixing on H . Then the restriction to H of any 1-cocycle w for σ is cohomologous to a cocycle which restricted to H is a character of H . If in addition we assume that either H is w -normal in Γ , or $\sigma|_H$ is mixing with H wq -normal in Γ , then $H^1(\sigma, \Gamma) = \text{Char}(\Gamma)$.*

Proof. Let $w \in Z^1(\sigma)$. Note that $w \times 1$ and $1 \times w$ are cocycles for the diagonal product action $\tilde{\sigma}$ of Γ on $L^\infty \overline{\otimes} L^\infty X$ and that $1 \otimes w$ is obtained by applying the flip automorphism on $L^\infty X \times L^\infty X$ to $w \otimes 1$. Thus, by Corollary 6.4.3 it follows that the restrictions to H of the cocycles $w \otimes 1, 1 \otimes w$ are equivalent in $Z^1(\tilde{\sigma}|_H)$. The Theorem then follows from the following general:

6.5.3. Proposition. *Let ρ be an action of an infinite group H on (X, μ) and $w \in Z^1(\rho)$. Assume $w \otimes 1 \sim 1 \otimes w$ in $Z^1(\tilde{\rho})$, where $\tilde{\rho}_h = \rho_h \otimes \rho_h, h \in H$. Then $w \sim 1$ in $Z^1(\rho)$.*

Proof. Let $u \in \mathcal{U}(L^\infty(X) \overline{\otimes} L^\infty X)$ be so that $(w_h \otimes 1)\tilde{\rho}_h(u) = u(1 \otimes w_h), h \in H$. For each $h \in H$ and $\tilde{\xi} \in L^2 X \overline{\otimes} L^2 X$, denote by $\tilde{\sigma}_h^w(\tilde{\xi}) = (w_h \otimes 1)\tilde{\sigma}_h(\tilde{\xi})(1 \otimes w_h)$. Notice that $\tilde{\sigma}_h^w$ is a unitary element on $L^2 X \overline{\otimes} L^2 X$ and that $h \mapsto \tilde{\sigma}_h$ is a unitary representation of H on the Hilbert space $L^2 X \overline{\otimes} L^2 X$.

Let G denote the connected component of id in $\tilde{\sigma}' \cap \text{Aut}(X \times X, \mu \times \mu)$. By the definition of malleability and Corollary 6.4.3 it follows that $\{\alpha(w^l) \mid l \in \mathbb{Z}\} \subset G$. Q.E.D.

6.5.4. Lemma. *Let Γ, L be discrete groups with Γ infinite. Let σ be a free, weakly mixing m.p. action of Γ on the probability space and β a free measure preserving action of L on the same probability space which commutes with σ . If $A^L \stackrel{\text{def}}{=} \{a \in A \mid \beta_h(a) = a, \forall h \in L\}$ then $\sigma_g(A^L) = A^L, \forall g \in \Gamma$, so $\sigma_g^L \stackrel{\text{def}}{=} \sigma_{g|_{A^L}}$ defines an integral preserving action of Γ on A^L .*

Proof. Since $\beta_h(\sigma_g(a)) = \sigma_g(\beta_h(a)) = \sigma_g(a), \forall h \in L, a \in A^L$, it follows that σ_g leaves A^L invariant $\forall g \in \Gamma$. Q.E.D.

6.5.5. Lemma. *With $\Gamma, L, \sigma, \beta, A^L, \sigma^L$ as in 6.5.4, assume the action σ^L of Γ on A^L is free. For each $\gamma \in \text{Char}(L)$ denote $\mathcal{U}_\gamma \stackrel{\text{def}}{=} \{v \in \mathcal{U}(A) \mid \beta_h(v) = \gamma(h)v, \forall h \in L\}$ and $\text{Char}_\beta(L) \stackrel{\text{def}}{=} \{\gamma \in \text{Char}(L) \mid \mathcal{U}_\gamma \neq \emptyset\}$. Then we have:*

1°. $\mathcal{U}_\gamma \mathcal{U}_{\gamma'} = \mathcal{U}_{\gamma\gamma'}, \forall \gamma, \gamma' \in \text{Char}(L)$, and $\text{Char}_\beta(L)$ is a countable group.

2°. If $\gamma_0 \in \text{Char}(\Gamma)$, $\gamma \in \text{Char}_\beta(L)$ and $v \in \mathcal{U}_\gamma$, then $w^{\gamma_0, \gamma}(g) \stackrel{\text{def}}{=} \sigma_g(v)v^*\gamma_0(g) \in A^L, \forall g \in \Gamma$, and $w^{\gamma_0, \gamma}$ defines a 1-cocycle for (σ^L, Γ) whose class in $H^1(\sigma^L, \Gamma)$ doesn't depend on the choice of $v \in \mathcal{U}_\gamma$.

Proof. 1°. If $v \in \mathcal{U}_\gamma, v' \in \mathcal{U}_{\gamma'}$ then

$$\beta_h(vv') = \beta_h(v)\beta_h(v') = \gamma(h)\gamma'(h)vv',$$

so $vv' \in \mathcal{U}_{\gamma\gamma'}$. This also implies $\text{Char}_\beta(L)$ is a group. Noticing that $\{\mathcal{U}_\gamma\}_\gamma$ are mutually orthogonal in $L^2(A, \tau) = L^2(X, \mu)$, by the separability of $L^2(X, \mu)$, $\text{Char}_\beta(L)$ follows countable.

2°. Since σ, β commute, $\sigma_g(\mathcal{U}_\gamma) = \mathcal{U}_\gamma, \forall g \in \Gamma, \gamma \in \text{Char}_\beta(L)$. In particular, $\sigma_g(v)v^* \in \mathcal{U}_1 = \mathcal{U}(A^L), \forall g \in \Gamma$ showing that the function $w^{\gamma_0, \gamma}$ takes values in $\mathcal{U}(A^L)$. Since $w^{\gamma_0, \gamma}$ is clearly a 1-cocycle for σ (in fact $w^{\gamma_0, \gamma} \sim_c \gamma_0 1$ as elements in $Z^1(\sigma, \Gamma)$), it follows that $w^{\gamma_0, \gamma} \in Z^1(\sigma^L)$.

If v' is another element in \mathcal{U}_γ then $u = v'v^* \in \mathcal{U}(A^L)$ and the associated 1-cocycles $w^{\gamma_0, \gamma}$ constructed out of v, v' follow cohomologous via u , in $Z^1(\sigma^L, \Gamma)$. Q.E.D.

6.5.6. Theorem. *Let $(\sigma, \Gamma), (\beta, L)$ be commuting, free m.p. actions on the same probability space, with Γ infinite and σ weakly mixing, as in 6.5.4, 6.5.5. Let $A^L, (\sigma^L, \Gamma)$ be defined as in 6.5.4 and $\text{Char}_\beta(L)$ as in 6.5.5. Also, for $\gamma_0 \in \text{Char}(L), \gamma \in \text{Char}_\beta(L)$ let $w^{\gamma_0, \gamma}$ be defined as in part 2° of 6.5.5. If $\text{Char}_\beta(L)$ is given the discrete topology then $\Delta : \text{Char}(\Gamma) \times \text{Char}_\beta(L) \rightarrow H^1(\sigma^L)$ defined by letting $\Delta(\gamma_0, \gamma)$ be the class of $w^{\gamma_0, \gamma}$ in $H^1(\sigma^L)$ is a 1 to 1 continuous group morphism. If in addition $H^1(\sigma) = \text{Char}(\Gamma)$ then Δ is an isomorphism of topological groups.*

Proof. The map Δ is clearly a group morphism and continuous. To see that it is 1 to 1 let $\gamma_0 \in \text{Char}(\Gamma), \gamma \in \text{Char}_\beta(L)$ and $v \in \mathcal{U}_\gamma$ and represent the element $\Delta(\gamma_0, \gamma) \in H^1(\sigma^L)$ by the 1-cocycle $w_g^{\gamma_0, \gamma} = \sigma_g(v)v^*\gamma_0(g), g \in \Gamma$. If $w^{\gamma_0, \gamma} \sim_c \mathbf{1}$ then there exists $u \in \mathcal{U}(A^L)$ such that $\sigma_g(u)u^* = \sigma_g(v)v^*\gamma_0(g), \forall g \in \Gamma$. Thus, if we denote $u_0 = uv^* \in \mathcal{U}(A)$ then $\sigma_g(u_0)u_0^* = \gamma_0(g)\mathbf{1}, \forall g$. It follows that $\sigma_g(\mathbb{C}u_0) = \mathbb{C}u_0, \forall g \in G$, and since σ is weakly mixing this implies $u_0 \in \mathbb{C}\mathbf{1}$ and $\gamma_0 = 1$. Thus, $v \in \mathbb{C}u \subset \mathcal{U}(A^L) = \mathcal{U}_1$, showing that $\gamma = 1$ as well.

If we assume $H^1(\sigma) = \text{Char}(\Gamma)$ and take $w \in Z^1(\sigma^L)$ then we can view w as a 1-cocycle for σ . But then $w \sim \gamma_0 1$, for some $\gamma_0 \in \text{Char}(\Gamma)$. Since σ is ergodic, there exists a unique $v \in \mathcal{U}(A)$ (up to multiplication by a scalar) such that $w_g = \sigma_g(v)v^*\gamma_0(g), \forall g \in \Gamma$. Since w is A^Γ -valued, $\sigma_g(v)v^* \in \mathcal{U}(A^\Gamma), \forall g$. Thus $\sigma_g(v)v^* = \beta_h(\sigma_g(v)v^*) = \sigma_g(\beta_h(v))\beta_h(v)^*, \forall g$. By the uniqueness of v this implies $\beta_h(v) = \gamma(h)v$, for some scalar $\gamma(h)$. The map $\Gamma \ni h \mapsto \gamma(h)$ is easily seen to be a character, so $w = w^{\gamma_0, \gamma}$ showing that $(\gamma_0, \gamma) \mapsto w^{\gamma_0, \gamma}$ is onto.

Since $H^1(\sigma) = \text{Char}(\Gamma)$ is compact, by ... and ... σ is strongly ergodic so σ^L is also strongly ergodic. Thus $H^1(\sigma^L)$ is Polish, with $\Delta(\text{Char}(\Gamma))$ a closed subgroup, implying

that $\Delta(\text{Char}_\beta(L)) \simeq \text{H}^1(\sigma^\Gamma)/\Delta(\text{Char}(\Gamma))$ is Polish. Since it is also countable, it is discrete. Thus, Δ is an isomorphism of topological groups. Q.E.D.

6.6. Some calculations of H^1 -groups.

6.6.1. Lemma. *Let Γ be an infinite group and σ be the Bernoulli shift action of Γ on $(X, \mu) = \Pi_g(\mathbb{T}, \lambda)_g$. With the notations of 6.5.4, 6.5.5, for any countable abelian group Λ there exists a countable abelian group L and a free action β of L on (X, μ) such that $\text{Char}_\beta(L) = \Lambda$, $[\sigma, \beta] = 0$ and $\sigma|_{A^L}$ is a free action of Γ . Moreover, if Λ is finite then one can take $L = \Lambda$ and β to be any action of $L = \Lambda$ on (X, μ) that commutes with σ and such that $\sigma \times \beta$ is a free action of $\Gamma \times L$.*

Proof. Let L be a countable dense subgroup in the (2'nd countable) compact group $\hat{\Lambda}$ and μ_0 be the Haar measure on $\hat{\Lambda}$. Let β_0 denote the action of L on $L^\infty(\hat{\Lambda}, \mu_0) = L(\Lambda)$ given by $\beta_0(h)(u_\gamma) = \gamma(h)u_\gamma, \forall h \in L$, where $\{u_\gamma\}_{\gamma \in \Lambda} \subset L(\Lambda)$ denotes the canonical basis of unitaries in the group von Neumann algebra $L(\Lambda)$ and $\gamma \in \Lambda$ is viewed as a character on $L \subset \hat{\Lambda}$. Denote $A_0 = L^\infty(\hat{\Lambda}, \mu_0) \overline{\otimes} L^\infty(\mathbb{T}, \lambda)$ and τ_0 the state on A_0 given by the product measure $\mu_0 \times \lambda$. Let β denote the product action of L on $\overline{\otimes}_{g \in \Gamma} (A_0, \tau_0)_g$ given by $\beta(h) = \otimes_g (\beta_0(h) \otimes id)_g$.

Since $(A_0, \tau_0) \simeq (L^\infty(\mathbb{T}, \lambda), \int \cdot d\lambda)$, we can view σ as the Bernoulli shift action of G on $A = \overline{\otimes}_g (A_0, \tau_0)_g$. By the construction of β we have $[\sigma, \beta] = 0$. Also, the fixed point algebra A^L contains a σ -invariant subalgebra on which σ acts as the (classic) Bernoulli shift. Thus, the restriction $\sigma^\Gamma = \sigma|_{A^L}$ is a free, mixing action of G . Finally, we see by construction that $\text{Char}_\beta(L) = \Lambda$.

The last part is trivial, once we notice that if the action $\sigma \times \beta$ of $\Gamma \times L$ on A is free then the action σ^L of Γ on A^L is free. Q.E.D.

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