285I Fall 2022: **Ergodic embeddings & approximate independence in II_1 factors. Applications.**
Instructor: Sorin Popa
Meetings: **MWF 3-4pm in MS7608**

The purpose of this class is to develop a series of II_1 factor analysis techniques and to use them towards approaching a broad range of problems in C*- and W*-algebras, notably free group factor problems, matrix-approximation of groups and tracial algebras, and bicentralizer phenomena.

A II_1 factor M is an infinite dimensional algebra/$\mathbb{C}$ endowed with a *-operation (analogue to conjugation of complex valued functions on $[0,1]$), with an operator norm $\| \ \|$ (analogue to the sup-norm $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$) and a positive trace state τ (analogue to the Lebesgue integral $\tau(x) = \int x(t)\mathrm{d}t$) with the property that the $\| \ \|$-unit ball of M is complete in the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}$, and which is simple, a requirement that imposes a high degree of non-commutativity.

These conditions insure that M acts as a weakly closed *-algebra of left multiplication operators on the Hilbert space $L^2(M, \tau)$, closed to polar and spectral decomposition, and calculus with Borel functions. They also insure that projections resulting from spectral calculus are unitary equivalent in M iff they have the same "dimension", as measured by the trace τ. This is like the matrix algebra $\mathbb{M}_n(\mathbb{C})$ endowed with the trace state $\tau = \frac{1}{n}Tr$, except that in a II_1 factor the dimension of projections can take any value in $[0,1]$ (the *continuous dimension* phenomenon).

Thus, a II_1 factor M can be viewed as a "continuous matrix algebra" a remarkable aspect that allows taking the "$t \times t$ matrix algebra" over M, or *t-amplification*, $M^t \overset{def}{=} \mathbb{M}_t(M)$, for any positive real number t.

II_1 factors arise naturally from geometric data, like infinite groups and their actions on spaces, groupoids and their representations on Hilbert space, and from quantum operations/deformations. These fascinating mathematical objects provide a unique environment where both randomness and rigidity phenomena live together, a co-existence that produces a large number of intriguing problems and striking results. We will focus on three central problems in this area:

(1) The *free group factor problems*, asking whether the II_1 factors $L\mathbb{F}_n$ associated with free groups of rank n, $2 \leq n \leq \infty$, are non-isomorphic and wether $L\mathbb{F}_\infty$ is ∞-generated (i.e., cannot be generated by finitely many elements).

(2) The problem of wether a given algebra, group, or group action, has *Connes approximate embedding* property, i.e., can be "simulated" in matrix algebras.

(3) Approximate bi-centralizer phenomena for states and II_1 factors M, in particular the so-called *Connes' bicentralizer* problem.

The specific techniques we will develop are the *incremental patching* and *iterative* methods for constructing (approximate) embeddings of abelian (more generally AFD) algebras into II_1 factors, under constraints; *Poisson-boundary* methods. We'll cover the papers listed below, but also material from other papers.

The prerequisite for this class is basic knowledge in operator algebras, notably II_1 factors, as for instance covered by the book [AP]. All registered students will get A, but will have to make assigned presentations related to these topics.

[AP] C. Anantharaman, S. Popa: *An introduction to* II_1 *factors*, https://www.math.ucla.edu/ popa/Books/IIunV15.pdf

[DPe] S. Das, J. Peterson: *Poisson boundaries of* II_1 *factors*, arXiv preprint 2020

[P1] S. Popa: *Independence properties in subalgebras of ultraproduct* II_1 *factors*, JFA **266** (2014), 5818-5846 (math.OA/1308.3982)

[P2] S. Popa: *Constructing MASAs with prescribed properties*, Kyoto J. of Math, 2019, **59** (2019), 367-397 (math.OA/1610.08945)

[P3] S. Popa: *Coarse decomposition of* II_1 *factors*, Duke Math. J. **170** (2021) 3073-3110 (math.OA/1811.09213)

[P4] S. Popa: *On ergodic embeddings of factors*, Communication in Math. Physics, **384** (2021), 971-996 (math.OA/1910.06923)