Math 170E
Introduction to Probability and Statistics 1: Probability

Midterm 2

Instructions: You have 45 minutes to complete this midterm. There are four problems, worth a total of 40 points. You are allowed to bring one cheat sheet, A4 or letter size. You can also print it or write it on a tablet and then print it. Only scientific calculators are allowed. For full credit, show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper. Do not forget to write your name, section, and UID in the space below.

Name: ______________________________
Student ID number: ______________________
Section: ____________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Problem 1.

Let Ω be a probability space endowed with a probability $\mathbb{P}$. Let $c \in \mathbb{R}$ and let $X \colon \Omega \to \mathbb{R}$ be a random variable supported on $[0, +\infty)$ with density $f_X \colon [0, +\infty) \to \mathbb{R}$ given by

$$\forall x \in [0, +\infty), \quad f_X(x) = \begin{cases} x, & \text{if } x \in [0, 1] \\ \frac{c}{x^3}, & \text{if } x \in (1, +\infty). \end{cases}$$

(a) [2 pts.] Find $c \in \mathbb{R}$ such that f_X defines a probability density function on $[0, +\infty)$.

Solution:
We have that

$$\int_0^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}x + c\int_1^{+\infty} \frac{1}{x^3}\,\mathrm{d}x = \frac{1}{2} + c \times \frac{1}{2},$$

so if we impose $\int_0^{+\infty} f_X(x)\,\mathrm{d}x = 1$, we get $c = 1$. We also remark that with that value, $f_X(x) \geqslant 0$ for every $x \geqslant 0$.

(b) [3 pts.] Compute $\mathbb{E}[X]$.

Solution: We have

$$\mathbb{E}[X] = \int_0^{+\infty} x f_X(x)\,\mathrm{d}x = \int_0^1 x \times x\,\mathrm{d}x + \int_1^{+\infty} x \times \frac{1}{x^3}\,\mathrm{d}x = \frac{1}{3} + 1 = \frac{4}{3}.$$

(c) [2 pts.] Find the median of X.

Solution: We remark that

$$\int_0^1 f_X(x)\,\mathrm{d}x = \frac{1}{2},$$

so $x_{0.5} = 1$.

(d) [3 pts.] Compute the 0.7-percentile $x_{0.7}$ of X.

Solution: Since $x_{0.5} = 1$, then $x_{0.7} > 1$, so we need to solve

$$0.7 = F(X_{0.7}) = \int_0^1 x\,\mathrm{d}x + \int_1^{x_{0.7}} \frac{1}{x^3}\,\mathrm{d}x = \frac{1}{2} + \left[-\frac{1}{2x^2}\right]_1^{x_{0.7}} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2x_{0.7}^2},$$

so

$$x_{0.7}^2 = \frac{1}{0.6} = \frac{5}{3}$$

which yields $x_{0.7} = \sqrt{\frac{5}{3}} \approx 1.29$.

Problem 2. *10 pts.*

Consider a discrete outcome space Ω endowed with a probability $\mathbb{P}$. Let $\lambda > 0$ and let $X : \Omega \to \mathbb{R}$ be a discrete random variable with $X \sim \text{Poi}(\lambda)$. Determine λ if

$$3\mathbb{P}(X = 1) + \mathbb{P}(X = 0) = \mathbb{P}(X = 2).$$

Solution: If $X \sim \text{Poi}(\lambda)$, then for any $k \in \mathbb{N}$,

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

Therefore

$$\mathbb{P}(X = 1) = e^{-\lambda}\lambda, \quad \mathbb{P}(X = 0) = e^{-\lambda}, \quad \mathbb{P}(X = 2) = e^{-\lambda}\frac{\lambda^2}{2},$$

so the equation is equivalent to

$$3e^{-\lambda}\lambda + e^{-\lambda} = e^{-\lambda}\frac{\lambda^2}{2}.$$

Since $e^{-\lambda} > 0$, we get

$$3\lambda + 1 = \frac{\lambda^2}{2}.$$

Solving this quadratic equation, we obtain

$$\lambda_1 = 3 - \sqrt{11}, \quad \lambda_2 = 3 + \sqrt{11}.$$

Since $\lambda_1 < 0$, the only possible solution is

$$\lambda = 3 + \sqrt{11}.$$

Problem 3.

A bag contains ten balls: five red, three yellow, and two white. Consider the following random experiment: pick four balls without replacement, and with all such selections being equally likely. Let X_1 denote the number of red balls selected and let X_2 denote the number of yellow balls selected.

(a) [3 pts.] Determine $\mathrm{supp}(X_1, X_2)$.

Solution: We have that

$$\mathrm{supp}(X_1, X_2) = \{(x_1, x_2) : x_1 \in \{0,1,2,3,4,5\}, x_2 \in \{0,1,2,3\}, (4-x_1-x_2) \in \{0,1,2\}\}.$$

Another possible answer is to provide the support in its explicit form:

$$\mathrm{supp}(X_1, X_2) = \{(0,2),(0,3),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(3,0),(3,1),(4,0)\}.$$

(b) [4 pts.] Find the joint density of (X_1, X_2).

Solution: We have that for any $(x_1, x_2) \in \mathrm{supp}(X_1, X_2)$,

$$f_{(X_1,X_2)}(x_1, x_2) = \frac{\binom{5}{x_1}\binom{3}{x_2}\binom{2}{4-x_1-x_2}}{\binom{10}{4}}.$$

(c) [3 pts.] Compute the marginal density f_{X_1}.

Solution: First, we remark that $\mathrm{supp}(X_1) = \{0,1,2,3,4\}$ since we draw no more than 4 balls in total. Then, for $x_1 \in \{0,1,2,3,4\}$, we have

$$f_{X_1}(x_1) = \frac{\binom{5}{x_1}\binom{5}{4-x_1}}{\binom{10}{4}}.$$

Problem 4.

Let Ω be a probability space endowed with a probability $\mathbb{P}$. Let $X\colon \Omega \to \mathbb{R}$ and $Y\colon \Omega \to \mathbb{R}$ be two continuous random variables supported on $\mathbb{R}$ and with densities $f_X\colon \mathbb{R} \to \mathbb{R}$ and $f_Y\colon \mathbb{R} \to \mathbb{R}$, respectively. We define the entropy of X, $\mathrm{H}(X)$, as

$$\mathrm{H}(X) \coloneqq -\int_{-\infty}^{+\infty} f_X(x)\log(f_X(x))\,\mathrm{d}x,$$

where log denotes the natural logarithm function. We also define the relative entropy between X and Y, denoted by $\mathrm{H}(X|Y)$, by

$$\mathrm{H}(X|Y) \coloneqq \int_{-\infty}^{+\infty} f_X(x)\log\left(\frac{f_X(x)}{f_Y(x)}\right)\mathrm{d}x,$$

which, as a matter of fact, is a nonnegative quantity. Let $\mu \in \mathbb{R}$ and $\sigma^2 > 0$.

(a) [4 pts.] Suppose $Y \sim \mathcal{N}(\mu, \sigma^2)$. Compute $\mathrm{H}(Y)$.

Solution:

$$\begin{aligned}
\mathrm{H}(Y) &= -\int_{-\infty}^{+\infty} \log(f_Y(x)) f_Y(x)\,\mathrm{d}x = -\int_{-\infty}^{+\infty} \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) f_Y(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty}\left[\log(\sqrt{2\pi\sigma^2}) + \frac{(x-\mu)^2}{2\sigma^2}\right] f_Y(x)\,\mathrm{d}x \\
&= \log(\sqrt{2\pi\sigma^2})\int_{-\infty}^{+\infty} f_Y(x)\,\mathrm{d}x + \frac{1}{2\sigma^2}\int_{-\infty}^{+\infty}(x-\mu)^2 f_Y(x)\,\mathrm{d}x \\
&= \log(\sqrt{2\pi\sigma^2}) \times 1 + \frac{1}{2\sigma^2} \times \sigma^2 \\
&= \log(\sqrt{2\pi\sigma^2}) + \frac{1}{2}.
\end{aligned}$$

(b) [6 pts.] Assume that $\mu_X = \mu$ and $\sigma_X^2 = \sigma^2$. Computing $\mathrm{H}(X|Y)$, prove that

$$\mathrm{H}(X) \leqslant \mathrm{H}(Y).$$

Solution: Let us compute $\mathrm{H}(X|Y)$:

$$\begin{aligned}
\mathrm{H}(X|Y) &= \int_{-\infty}^{+\infty} f_X(x) \log\left(\frac{f_X(x)}{f_Y(x)}\right) \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} f_X(x) \log(f_X(x)) \,\mathrm{d}x - \int_{-\infty}^{+\infty} f_X(x) \log(f_Y(x)) \,\mathrm{d}x \\
&= -\mathrm{H}(X) + \int_{-\infty}^{+\infty} \left[\log(\sqrt{2\pi\sigma^2}) + \frac{(x-\mu)^2}{2\sigma^2}\right] f_X(x) \,\mathrm{d}x \\
&= -\mathrm{H}(X) + \log(\sqrt{2\pi\sigma^2}) \int_{-\infty}^{+\infty} f_X(x) \,\mathrm{d}x + \frac{1}{2\sigma^2} \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x) \,\mathrm{d}x \\
&= -\mathrm{H}(X) + \log(\sqrt{2\pi\sigma^2}) \times 1 + \frac{1}{2\sigma^2} \times \sigma_X^2 \\
&= -\mathrm{H}(X) + \log(\sqrt{2\pi\sigma^2}) + \frac{1}{2} \\
&= -\mathrm{H}(X) + \mathrm{H}(Y).
\end{aligned}$$

Since $\mathrm{H}(X|Y) \geqslant 0$, we deduce that

$$\mathrm{H}(X) \leqslant \mathrm{H}(Y).$$