# Hausdorff Dimension of Continuous Images

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**Effective descriptive set theory:**

Computability theory $\approx$ Descriptive set theory

More recently:

Algorithmic randomness $\approx$ Geometric measure theory

There are many theorems in algorithmic randomness proved before this connection was fully understood.

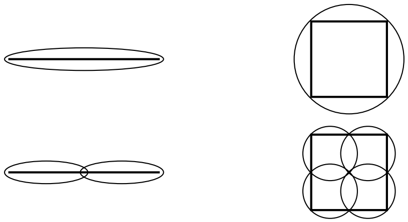
There are many theorems in geometric measure theory proved before this connection was fully understood.

Question. Do any of these theorems yield new results on the other side of the correspondence?

**This talk.** One example of this

# Hausdorff Dimension

**Idea of Hausdorff dimension:** Measure the dimension $A \subseteq \mathbb{R}^n$ by looking at how the number of balls of radius $r$ needed to cover $A$ scales as $r \to 0$.



**Unit interval:** $r$ halved $\rightsquigarrow$ number of balls needed doubles

**Unit square:** $r$ halved $\rightsquigarrow$ number of balls needed quadruples

**In general:** A set of dimension $d$ needs $\sim (1/r)^d$ balls of radius $r$

Hausdorff dimension: Take this as the definition of "A has dimension d"

Actually, this is more like Minkowski dimension. Hausdorff dimension is defined to better handle "irregular" sets.

Definition. A set $A$ in a metric space has measure 0 in dimension $d$ if for all $\epsilon > 0$, there is a countable collection of balls $\{B_n\}_{n\in\omega}$ with radii $\{r_n\}_{n\in\omega}$ such that

- $A \subseteq \bigcup_n B_n$
- and $\sum_{n\in\omega} r_n^d < \epsilon$

Definition. The Hausdorff dimension of $A$, denoted $\dim(A)$, is

$$\dim(A) = \inf\{d \mid A \text{ has measure 0 in dimension } d\}.$$

**Intuition:** Suppose that for every $r$, $A$ can be covered with $(1/r)^d$ balls of radius $r$

For any $d' > d$,

$$(1/r)^d \cdot r^{d'} = r^{d'-d}$$

which goes to 0 as $r \to 0$. So $A$ has measure 0 in dimension $d'$.

# Effective Hausdorff Dimension

**Definition.** For a finite string $\sigma \in 2^{<\omega}$, the Kolmogorov complexity of $\sigma$, denoted $C(\sigma)$, is the length of the shortest program that outputs $\sigma$.

**Definition.** For $x \in 2^{\omega}$, the effective Hausdorff dimension of $x$, denoted $\dim(x)$, is

$$\dim(x) = \liminf_{n\to\infty} \frac{C(x\upharpoonright n)}{n}.$$

**Informally:** $\dim(x) \approx$ number of bits needed to describe the first $n$ bits of $x$ as a fraction of $n$

**Example.** Let $x$ be a sequence such that all even bits are 0 and all odd bits are chosen by flipping a coin.

$$x = 0100000101010001000100010000000010\ldots$$

For any $n$, $C(x\upharpoonright n) \approx n/2 \implies \dim(x) = 1/2$.

**Comment:** All the definitions can also be relativized. For any $a \in 2^{\omega}$, we can define $C^{a}(\sigma)$ and $\dim^{a}(\sigma)$.

## How is effective Hausdorff dimension similar to Hausdorff dimension?

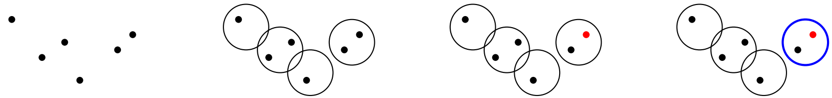
**Another way to view Hausdorff dimension:** You are playing a game with your friend using the set $A$:

(1) First you pick an arbitrary point $x \in A$ and an arbitrary $r > 0$

(2) Your goal is to describe $x$ to your friend as concisely as possible

(3) More precisely: you need to give some information to your friend to allow them to guess a point that is within distance $r$ of $x$ and you want to give as little information as possible

**The point:** If $A$ can be covered by $(1/r)^d$ balls of radius $r$, you only need to give your friend $\log((1/r)^d) = d\log(1/r)$ bits of information



If $A \subseteq 2^\omega$ then guessing $x$ within distance $2^{-n}$ corresponds to guessing the first $n$ bits of $x$, which we can describe using at most $dn$ bits

# The Point-to-Set Principle

The connection between effective Hausdorff dimension and Hausdorff dimension is more than just conceptual.

**Theorem (J. Lutz and N. Lutz).** For any set $A \subseteq 2^\omega$

$$\dim(A) = \min_a \sup_{x \in A} \dim^a(x).$$

**Idea of the proof.**
$\geq$ Roughly the idea on the previous slide
$\leq$ For the appropriate $a$, $A \subseteq \{x \mid \dim^a(x) \leq d\}$, which can be proved to have dimension $d$

**Idea:** Translate theorems from algorithmic randomness into geometric measure theory (or vice-versa)

- Replace $x \in 2^\omega$ with $A \subseteq \mathbb{R}^n$ (or $2^\omega$)
- Replace $\dim(x)$ with $\dim(A)$

Can be extended by the usual dictionary of effective descriptive set theory. E.g.

- Replace computable with continuous

# Case study: Miller's theorem

**Idea:** Translate theorems from algorithmic randomness into geometric measure theory (or vice-versa).

**Prominent early question in algorithmic randomness:** Does every $x$ with $0 < \dim(x) < 1$ compute some $y$ with $\dim(y) > \dim(x)$?

Answered in 2011:

Theorem (J. Miller). There is some $x \in 2^\omega$ such that $\dim(x) = 1/2$ and for all $y \leq_T x$, $\dim(y) \leq 1/2$. (Can replace $1/2$ with any computable $d \in [0, 1]$)

**Informal translation of Miller's theorem:** There is some $A \subseteq \mathbb{R}$ such that $\dim(A) = 1/2$ and for all continuous $f : \mathbb{R} \to \mathbb{R}$, $\dim(f(A)) \leq 1/2$.

Questions.

(1) Is this true?

(2) Is this interesting?

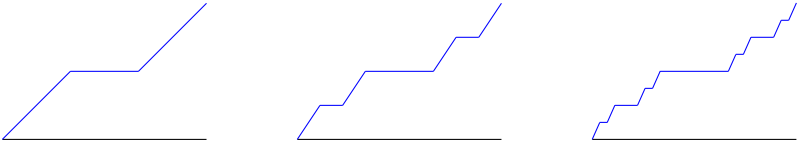
**Informal translation of Miller's theorem:** There is some $A \subseteq \mathbb{R}$ such that $\dim(A) = 1/2$ and for all continuous $f \colon \mathbb{R} \to \mathbb{R}$, $\dim(f(A)) \leq 1/2$.

Question. If true, is this interesting?

It is usually easy to increase Hausdorff dimension!

Example 1. Cantor middle-thirds set, $C$.

There is a continuous surjection $f \colon C \to [0, 1]$.



$\dim(C) = \log_3(2)$

$\dim(f(C)) = \dim([0, 1]) = 1.$

**Informal translation of Miller's theorem:** There is some $A \subseteq \mathbb{R}$ such that $\dim(A) = 1/2$ and for all continuous $f: \mathbb{R} \to \mathbb{R}$, $\dim(f(A)) \leq 1/2$.

Question. If true, is this interesting?

It is usually easy to increase Hausdorff dimension!

Example 2. Some facts from geometric measure theory

- Marstrand's projection theorem. For any analytic set $A \subseteq \mathbb{R}^2$, if $\dim(A) \geq 1$ then for a random linear projection $p: \mathbb{R}^2 \to \mathbb{R}$, $\dim(p(A)) = 1$.
- Theorem (Kaufman). For any set $A \subseteq \mathbb{R}$, Brownian motion in the plane, $B$, almost surely sends $A$ to a set $B(A) \subseteq \mathbb{R}^2$ of dimension $\dim(B(A)) = 2\dim(A)$.

$\mathbb{R}^2$ $\qquad \dim(B(A)) = 1$

$\mathbb{R}$ $\qquad \dim(A) = 1/2 \nearrow \qquad \searrow \dim(p(B(A))) = 1$

**Informal translation of Miller's theorem:** There is some $A \subseteq \mathbb{R}$ such that $\dim(A) = 1/2$ and for all continuous $f : \mathbb{R} \to \mathbb{R}$, $\dim(f(A)) \leq 1/2$.

Question. If true, is this interesting?

It is usually easy to increase Hausdorff dimension!

Example 3. Descriptive set theory hammer.

- Perfect set theorem. Every analytic set is either countable or contains a perfect set.
- Folklore. If $A$ is a perfect set then there is a continuous surjection $f : A \to [0, 1]$ (which can be extended to all of $\mathbb{R}$ by Tietze's extension theorem).

So if $A \subseteq \mathbb{R}$ is analytic and uncountable there is a continuous $f : \mathbb{R} \to [0, 1]$ such that $f(A) = [0, 1]$.

$\implies \dim(f(A)) = \dim([0, 1]) = 1$

**Informal translation of Miller's theorem:** There is some $A \subseteq \mathbb{R}$ such that $\dim(A) = 1/2$ and for all continuous $f : \mathbb{R} \to \mathbb{R}$, $\dim(f(A)) \leq 1/2$.

Question. Is this true?

Answer. No, assuming AD.

Answer. Yes, assuming CH.

Theorem (L. and Miller). Assuming CH, there is some $A \subseteq \mathbb{R}$ such that $\dim(A) = 1/2$ and for all continuous functions $f : \mathbb{R} \to \mathbb{R}$, $\dim(f(A)) \leq 1/2$. Also works for any $d \in [0, 1]$ in place of $1/2$

It seems natural to use the point-to-set principle + Miller's theorem. But that does not quite work.

Instead, we need to first strengthen Miller's theorem.

# How to prove it

Theorem (L. and Miller). Assuming CH, there is some $A \subseteq 2^\omega$ such that $\dim(A) = 1/2$ and for all continuous functions $f\colon 2^\omega \to 2^\omega$, $\dim(f(A)) \leq 1/2$.

Miller's theorem. There is some $x \in 2^\omega$ such that $\dim(x) = 1/2$ and for all $y \leq_T x$, $\dim(y) \leq 1/2$.

Miller's theorem, relativized. For all $a \in 2^\omega$, there is some $x \in 2^\omega$ such that $\dim^a(x) = 1/2$ and for all $y \leq_T x \oplus a$, $\dim^a(y) \leq 1/2$.

What we need. For every countable sequence $\{a_n\}_{n\in\omega}$ of elements of $2^\omega$, there is some $x \in 2^\omega$ such that for every $n$,

- $\dim^{a_n}(x) \geq 1/2$
- for all $y \leq_T x \oplus a_n$, $\dim^{a_n}(y) \leq 1/2$.

In other words, $x$ witnesses Miller's theorem relatively to countably many oracles simultaneously.

($\star$) *For every countable sequence $\{a_n\}_{n\in\omega}$ of elements of $2^\omega$, there is some $x \in 2^\omega$ such that for every $n$,*

- $\dim^{a_n}(x) \geq 1/2$
- *for all $y \leq_T x \oplus a_n$, $\dim^{a_n}(y) \leq 1/2$.*

Proof of theorem using ($\star$). Fix an enumeration $\{a_\alpha\}_{\alpha<\omega_1}$ of $2^\omega$.

For each $\alpha$, choose

$$x_\alpha \text{ witness to } (\star) \text{ applied to } \{a_\beta\}_{\beta<\alpha}$$

and set $A = \{x_\alpha \mid \alpha < \omega_1\}$.

For any $a_\alpha$, $\dim^{a_\alpha}(x_{\alpha+1}) \geq 1/2$ so $\dim(A) \geq 1/2$.

For any $f\colon 2^\omega \to 2^\omega$ continuous, $f$ is computable relative to some $a_\alpha$.

$$f(A) = \underbrace{\{f(x_\beta) \mid \beta \leq \alpha\}}_{\text{countable}} \quad \cup \quad \underbrace{\{f(x_\beta) \mid \beta > \alpha\}}_{\text{dimension at most } 1/2 \text{ as witnessed by } a_\alpha}$$

So $\dim(f(A)) \leq 1/2$.

What we need. For every countable sequence $\{a_n\}_{n\in\omega}$ of elements of $2^\omega$, there is some $x \in 2^\omega$ such that for every $n$,

- $\dim^{a_n}(x) \geq 1/2$
- for all $y \leq_T x \oplus a_n$, $\dim^{a_n}(y) \leq 1/2$.

Actually, I don't know if this is true. But an easier statement is enough for the proof.

~~$f : 2^\omega \to 2^\omega$ is continuous $\approx$ $f$ is computable~~

$f : 2^\omega \to 2^\omega$ is continuous $\approx$ $f$ is truth-table computable

Revised statement. For every countable sequence $\{a_n\}_{n\in\omega}$ of elements of $2^\omega$, there is some $x \in 2^\omega$ such that for every $n$,

- $\dim^{a_n}(x) \geq 1/2$
- for all $y \leq_{tt}^{a_n} x$, $\dim^{a_n}(y) \leq 1/2$.

Still not that easy to prove. But can be proved using ideas somewhat similar to those used in the proof of Miller's theorem.

# Extension to $\mathbb{R}^n$

Not hard to modify the proof to work in $\mathbb{R}^n$:

Theorem (L. and Miller). Assuming CH, for any $d \in [0,1]$ and $n \in \mathbb{N}$, there is some $A \subseteq \mathbb{R}^n$ such that $\dim(A) = dn$ and for all continuous functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $\dim(f(A)) \leq dm$.

In some cases, even more is possible.

Theorem (L. and Miller). Assuming CH, there is some $A \subseteq \mathbb{R}^2$ such that $\dim(A) = 1$ and for all continuous functions $f\colon \mathbb{R}^2 \to \mathbb{R}$, $\dim(f(A)) = 0$.

The analogous statement for $2^\omega$ is false.

The digit interleaving map $2^\omega \times 2^\omega \to 2^\omega$

$$(x_0 x_1 x_2 \ldots, y_0 y_1 y_2 \ldots) \mapsto x_0 y_0 x_1 y_1 x_2 y_2 \ldots$$

preserves relative dimension.

Theorem (L. and Miller). Assuming CH, there is some $A \subseteq \mathbb{R}^2$ such that $\dim(A) = 1$ and for all continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$, $\dim(f(A)) = 0$.

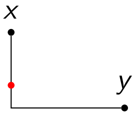
How is this possible? It helps to first consider an easier statement.

Proposition. If $f : \mathbb{R}^2 \to \mathbb{R}$ is computable then there is $x \in \mathbb{R}^2$ such that $x$ is not computable and $f(x)$ is.

Proof. Two cases:

*Case 1.* For all $x, y \in \mathbb{R}^2$ with both coordinates noncomputable, $f(x) = f(y)$. $\implies$ $f$ is constant.

*Case 2.* For some $x, y \in \mathbb{R}^2$ with both coordinates noncomputable, $f(x) \neq f(y)$.




Key fact. If $A \subseteq \mathbb{R}^2$ has $\dim(A) < 1$ then $\mathbb{R}^2 \setminus A$ contains a connected set of positive measure.

Corollary. If $f : \mathbb{R}^2 \to \mathbb{R}$ is computable then there is some $x \in \mathbb{R}^2 \setminus A$ such that $f(x)$ is computable.

Proof of corollary. Suppose $B \subseteq \mathbb{R}^2 \setminus A$ connected and positive measure.

*Case 1.* $f$ is constant on $B \implies f(B)$ is computable

*Case 2.* $f$ is not constant on $B \implies$ for some $x \in B$, $f(x)$ is rational

Key fact. If $A \subseteq \mathbb{R}^2$ has $\dim(A) < 1$ then $\mathbb{R}^2 \setminus A$ contains a connected set of positive measure.

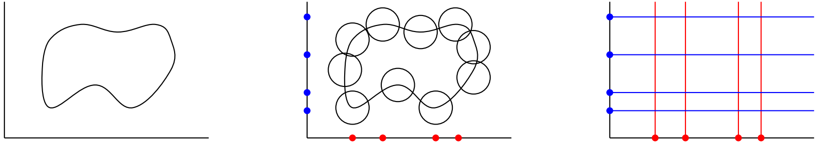
Corollary. If $f : \mathbb{R}^2 \to \mathbb{R}$ is computable then there is some $x \in \mathbb{R}^2 \setminus A$ such that $f(x)$ is computable.

Proof of key fact. $\dim(A) < 1 \implies A$ can be covered with balls whose diameters sum to less than 1

So the complement of the projection of $A$ onto the $x$-axis has positive measure. Call this set $B_1$

and the complement of the projection of $A$ onto the $y$-axis has positive measure. Call this set $B_2$

Then $(B_1 \times [0,1]) \cup ([0,1] \times B_2) \subseteq \mathbb{R}^2 \setminus A$ is connected and positive measure



**Some questions.**

Question 1. Can the result on preserving dimension be strengthened to hold for all Borel functions?

If so, likely requires strengthening Miller's theorem.

Question 2. Find more examples where theorems/definitions/etc in algorithmic randomness translate into new theorems/definitions/etc in geometric measure theory (or vice-versa).

A possible example?

Hausdorff measure $\approx$ a priori Kolmogorov complexity

(Hard) theorem of Gács and Day: a priori complexity and monotone complexity are not the same

Does monotone complexity correspond to anything in geometric measure theory? If so, does the Gács-Day theorem have some nice interpretation?