# ① Introduction

Gödel's completeness theorem   consistent axioms ⇝ model

**Question** computable consistent axioms ⇝ computable model?

**More formally** Does every consistent, c.e. theory have a computable model?

domain is $\mathbb{N}$ and all constants, functions, relations uniformly computable

**Answer** No!

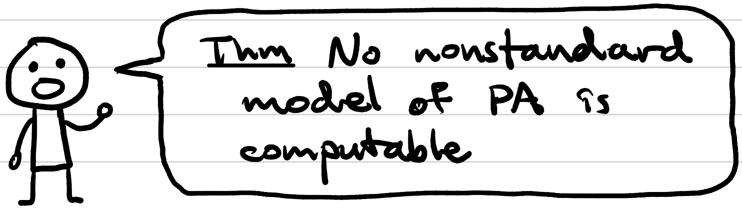
Method 1: Direct construction

E.g. $T$ describes a path through a computable infinite binary tree with no computable paths

Method 2: Tennenbaum's theorem

E.g. $PA + \neg Con(PA)$
ZFC
$RCA_0$

## (1.1) Tennenbaum's theorem




Stan Tennenbaum

Tennenbaum's theorem gives many examples of consistent, c.e. theories with no computable models

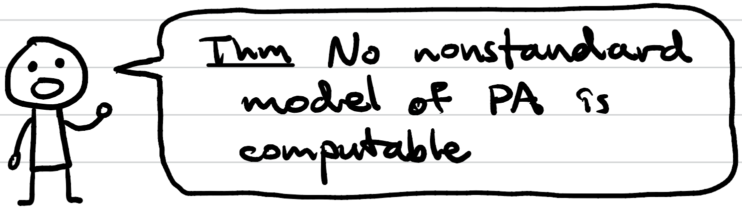
Examples $PA + \neg Con(PA)$

$\mathbb{N} \not\models PA + \neg Con(PA) \Rightarrow$ All models of $PA + \neg Con(PA)$ are nonstandard

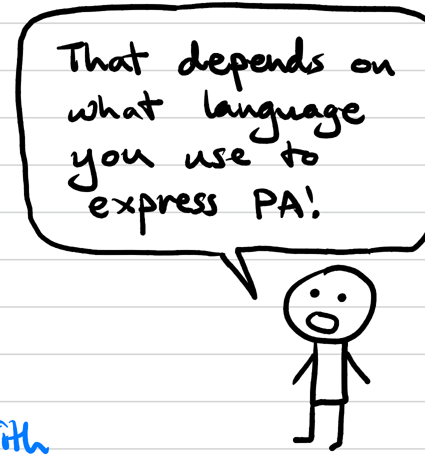
Tennenbaum's thm $\Rightarrow$ No computable models

$ZFC$, $ZF$, $RCA_0$, etc. By adapting the proof

## 1.2 Pakhomov's theorem




Stan Tennenbaum




Fedor Pakhomov

Key notion: Definitional equivalence

$T \approx T'$ if they are the same thy, but with different choice of what concepts to take as primitive

↳ A strong form of bi-interpretability

Pakhomov's theorem, informal version: For every thy we listed on the previous slide, there is a definitionally equivalent theory with a computable model

E.g. $PA + \neg Con(PA)$, ZFC, etc.

Every $\mathcal{L}'$-symbol has an $\mathcal{L}$-definition

**Def** $T \subseteq T'$, $\mathcal{L} \subseteq \mathcal{L}'$ is a definitional extension if:

1. $T'$ is conservative over $T$
2. For every constant symbol $c \in \mathcal{L}' \setminus \mathcal{L}$ there is an $\mathcal{L}$-formula $\varphi_c(x)$ s.t. $T' \vdash \forall x\,(\varphi_c(x) \leftrightarrow x = c)$
3. Similarly for every relation & function symbol $\in \mathcal{L}' \setminus \mathcal{L}$

**Example** Adding emptyset symbol to ZFC

$\mathcal{L} = \{\in\}$, $\mathcal{L}' = \{\in, \emptyset\}$, $T = ZFC$, $T' = ZFC + \forall x\,(x \notin \emptyset)$

w/ disjoint signatures

**Def** Theories $T$ & $T'$ in languages $\mathcal{L}, \mathcal{L}'$ are definitionally equivalent if they have a common definitional extension

**Example** $T = Th(\mathbb{Z}, +)$, $T' = Th(\mathbb{Z}, -)$, $T'' = Th(\mathbb{Z}, +, -)$

$x + y = z \iff x = z - y \qquad x - y = z \iff x = z + y$

**Key pt** If $T, T'$ def. equiv. and $M \models T$ then you can also view $M$ as a model of $T'$

↳ by interpreting each symbol of $T'$ by its $\mathcal{L}$-definition

**Def** $T \subseteq T'$, $\mathcal{L} \subseteq \mathcal{L}'$ is a definitional extension if:

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3. Similarly for every relation & function symbol $\in \mathcal{L}' \setminus \mathcal{L}$

**Def** Theories $T$ & $T'$ in languages $\mathcal{L}, \mathcal{L}'$ are definitionally equivalent if they have a common definitional extension

**Thm** (Pakhomov) There is a thy $T$ definitionally equivalent to PA s.t. every consistent, c.e. extension of $T$ has a computable model

$\Rightarrow$ $\exists T \approx PA + \neg Con(PA)$, $T$ has a computable model

Same proof $\Rightarrow$ $\exists T \approx ZFC$, $T$ has a computable model

Seems like it should also work for $RCA_0$, etc.

## (1.3) Pakhomov's question

Gödel's completeness theorem consistent axioms $\rightsquigarrow$ model

**Question** computable consistent axioms $\rightsquigarrow$ computable model?

**Answer** No! Method 1: Direct construction Method 2: Tennenbaum's theorem

**Question** computable consistent axioms $\rightsquigarrow$ computable model of a def. equiv. thy?

Pakhomov: Tennenbaum's theorem no longer gives examples

**Answer** No! Method 1 still works (though it is harder)

Focus of the rest of this talk

**Thm** (L. & Walsh) There is a consistent, c.e. theory T such that no theory definitionally equivalent to T has a computable model

# A theory which really doesn't have a computable model

Based on the paper "A theory satisfying a strong version of Tennenbaum's theorem" with James Walsh.

## ② Proof strategy

**Thm** (L. & Walsh) There is a consistent, c.e. theory $T$ such that no theory definitionally equivalent to $T$ has a computable model

**Idea** Build a theory $T$ such that

1. $T$ has no computable models
2. Any theory def. equiv. to $T$ is model-theoretically tame

**Why is this useful?** Suppose $T$ has no computable models, $T'$ is def. equiv. to $T$ and $M \models T'$

$M$ can be seen as a model of $T$ in a definable way

$M$ has QE $\Rightarrow$ definitions are quantifier-free
$\Rightarrow$ $M$ computes a model of $T$
$\Rightarrow$ $M$ is not computable

Idea Build a theory T such that

1. T has no computable models
2. Any theory def. equiv. to T is model-theoretically tame

T' has QE ⇒ Every model of T' computes a model of T

Key tool Laskowski's theory of mutual algebraicity

T mutually algebraic ⇒ T' mutually algebraic
⇒ T' has a weak form of QE

Problem Don't know how to get full QE

Solution T has weak QE ⇒ Every model of T' computably approximates a model of T

Build T so that no model is computably approximable

## ③ The theory

**Def** Given $f: \mathbb{N} \to \mathbb{N}$, $x \in 2^\omega$ is f-guessable if there is an algorithm which, for every $n$, enumerates a list of at most $O(f(n))$ strings, one of which is $x \upharpoonright n$.

**Example** $y \in 2^\omega$ arbitrary
$x = 0y_0 00y_1 0000y_2 00000000y_3 00\ldots$ is $n$-guessable

**Prop** There is a computable, infinite binary tree $R$ such that no infinite path through $R$ is $n^2$-guessable

Can build $R$ directly or take $R$ to be a computable tree whose paths are all Martin-Löf random

For the rest of this talk, fix one such $R$

Essentially, $T$ is the simplest theory all of whose models code a path through $R$

The language

1. Constant $0$
2. Unary functions $S$, $P$
3. Unary relation $A$

Notation

1. $x + \underline{n} = \overbrace{S(S(\cdots S}^{n}(x)\cdots))$ $\quad x + \underline{3} = S(S(S(x)))$
2. $x - \underline{n} = P(P(\cdots P(x)\cdots))$ $\quad x - \underline{2} = P(P(x))$
3. $\underline{n} = 0 + \underline{n}$, $\underline{-n} = 0 - \underline{n}$

The theory

1. Theory of the integers with predecessor and successor $\mathrm{Th}(\mathbb{Z}, 0, x \mapsto x+1, x \mapsto x-1)$
2. $A$ codes a path through $R$

For all $n$,

$$\bigvee_{\sigma \in R_n} \left( \bigwedge_{\sigma(i)=1} A(\underline{i}) \wedge \bigwedge_{\sigma(i)=0} \neg A(\underline{i}) \right)$$

where $R_n = \{\sigma \in R \mid |\sigma| = n\}$

Key point If $M \models T$ then $A(\underline{0}), A(\underline{1}), A(\underline{2}), \ldots$ codes a path through $R$

| | | | |
|---|---|---|---|
| T | F | F | ... |
| 1 | 0 | 0 | ... $\in [R]$ |

## 3.1 Models of T

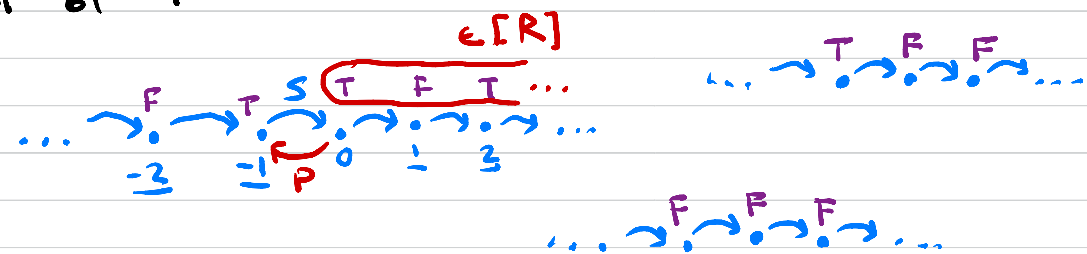
$\mathcal{R}$    Computable tree with no $n^2$-guessable paths

$\mathcal{L}$    $0, S, P, A$

$T$    ① $\mathrm{Th}(\mathbb{Z}, 0, x \mapsto x+1, x \mapsto x-1)$

② For all $n$, $\bigvee_{\sigma \in R_n} \left( \bigwedge_{\sigma(i)=1} A(\underline{i}) \wedge \bigwedge_{\sigma(i)=0} \neg A(\underline{i}) \right)$

Model of $T$




<u>Def</u> Given $M \models T$ and $a, b \in M$ the **distance between a and b** is the unique $k \in \mathbb{N}$ s.t. $a = b + \underline{k}$ or $b = a + \underline{k}$ or $\infty$ if no such $k$ exists

Originally due to Goncharov, Harizanov, Laskowski, Lempp, McCoy
Extensively developed by Laskowski

## ④ Mutual algebraicity

**Def** Given a model $M$, a formula $\varphi(\bar{x})$ is mutually algebraic over $M$ if there is $K \in \mathbb{N}$ such that for every nontrivial partition $\bar{x} = \bar{x}_0 \cup \bar{x}_1$ and every $\bar{a} \in M$

$$|\{\bar{b} \in M \mid M \models \varphi(\bar{a}, \bar{b})\}| \leq K$$

Example $M = (\mathbb{Z}, +)$
$x = y + 5$ is mutually algebraic
$x = y + z + 5$ is not

**Def** $M$ is mutually algebraic if every formula is equivalent to a Boolean combination of formulas which are mutually algebraic over $M$

Example $(\mathbb{Z}, x \mapsto x+1)$ is mutually alg.
$(\mathbb{Q}, \leq)$ is not (despite QE)
$x \leq y$ not equivalent to a Boolean comb. of mut. alg. formulas

## 4.1 Key facts

**Prop** Every model of our theory $T$ is mutually algebraic

pf QE + atomic formulas mut. alg.

Mutual algebraicity is preserved by definitional equivalence

**Prop** If $T, T'$ are definitionally equivalent and every model of $T$ is mutually algebraic then the same holds for $T'$

pf Mutual alg. only depends on the algebra of definable sets

Mutually alg. structures have a weak form of QE

**Thm** (essentially Laskowski) If $M$ is mutually algebraic then for every mutually algebraic formula $\varphi(\bar{x})$ there is a mutually algebraic formula $\psi(\bar{x}) = \exists \bar{y}\, \theta(\bar{x}, \bar{y})$ s.t. → possibly w/ parameters from $M$

1. $M \models \forall \bar{x} \, (\varphi(\bar{x}) \to \psi(\bar{x}))$
2. $\theta(\bar{x}, \bar{y})$ is quantifier free

## ⑤ The proof (sort of)

$R$ Computable tree with no $n^2$-guessable paths

$\mathcal{L}$ $0, S, P, A$

$T$ ① $\mathrm{Th}(\mathbb{Z}, 0, x \mapsto x+1, x \mapsto x-1)$

② For all $n$, $\bigvee_{\sigma \in R_n} \left( \bigwedge_{\sigma(i)=1} A(\underline{i}) \wedge \bigwedge_{\sigma(i)=0} \neg A(\underline{i}) \right)$

$\hookrightarrow$ $A(\underline{0}), A(\underline{1}), A(\underline{2}), \ldots$ codes a path through $R$

Fix $T', \mathcal{L}'$ def. equivalent to $T$ $\rightarrow$ Can assume $\mathcal{L}'$ has finite signature

$M$ a model of $T'$

Recall that we can view $M$ as a model of $T$

So it makes sense to talk about the truth values of $A(\underline{0}), A(\underline{1}), A(\underline{2}), \ldots$ in $M$

Strategy: Show that this sequence is $n^2$-guessable relative to an oracle for $M$

$T'$, $\mathcal{L}'$ def. equivalent to $T$
$M$ a model of $T'$

Recall that we can view $M$ as a model of $T$
So it makes sense to talk about the truth values
of $A(\underline{0})$, $A(\underline{1})$, $A(\underline{2})$, ... in $M$

Strategy Show that this sequence is $n^2$-guessable
relative to an oracle for $M$

Three steps
① Algorithm for guessing successors & predecessors
② Algorithm for guessing neighborhoods
Given $a$, $n$ guess $a-\underline{n}, \ldots, a, \ldots, a+\underline{n}$
③ Algorithm for guessing $A(\underline{0})$, $A(\underline{1})$, $A(\underline{2})$, ...

## 5.1 Guessing successors & predecessors

**Prop** There is an algorithm using an oracle for $M$ which, given any $a \in M$ enumerates $O(1)$ guesses for $S(a)$ and $O(1)$ guesses for $P(a)$, with both lists containing the correct value

**pf** $\varphi_S(x,y)$ $\Sigma'$-def. of $S$ $\quad M \models S(x) = y \leftrightarrow \varphi_S(x,y)$

Weak QE $\Rightarrow$ $\psi_S(x,y) = \exists \bar{z}\, \theta_S(x,y,\bar{z})$ $\leftarrow$ mut. alg.

QF

s.t. $M \models \varphi_S(x,y) \to \psi_S(x,y)$

Candidates for $S(a)$: $\{b \mid M \models \psi_S(a,b)\}$

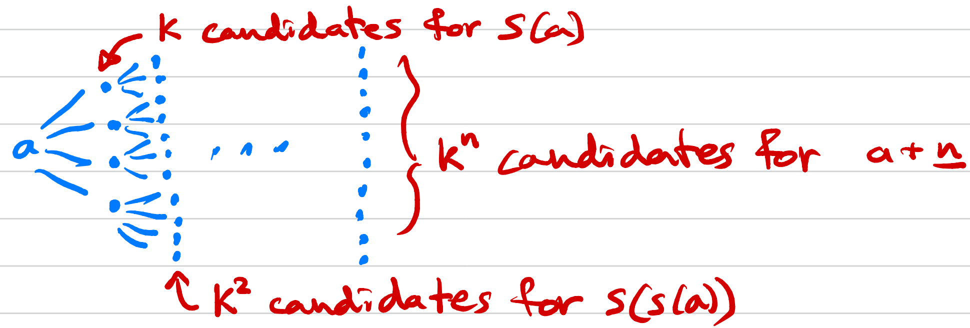
1. Enumerable $\quad \psi_S$ is existential
2. Includes $S(a)$ $\quad S(a) = b \Rightarrow M \models \varphi_S(a,b) \Rightarrow M \models \psi_S(a,b)$
3. Bounded size $\quad \psi_S$ mut. alg.

Candidates for $P(a)$: $\{b \mid M \models \psi_S(b,a)\}$

## 5.2 Guessing neighborhoods

**Prop** There is an algorithm which given $a \in M$ and $n \in \mathbb{N}$, enumerates $O(n^2)$ guesses for the sequence $a, a+\underline{1}, a+\underline{2}, \ldots, a+\underline{n}$, at least one of which is correct

**Idea**




**Problem** $\approx k^n$ guesses $\gg n^2$

**Solution** Mutual alg. to the rescue!
Can show all candidates for $s(a)$ are a short distance from a

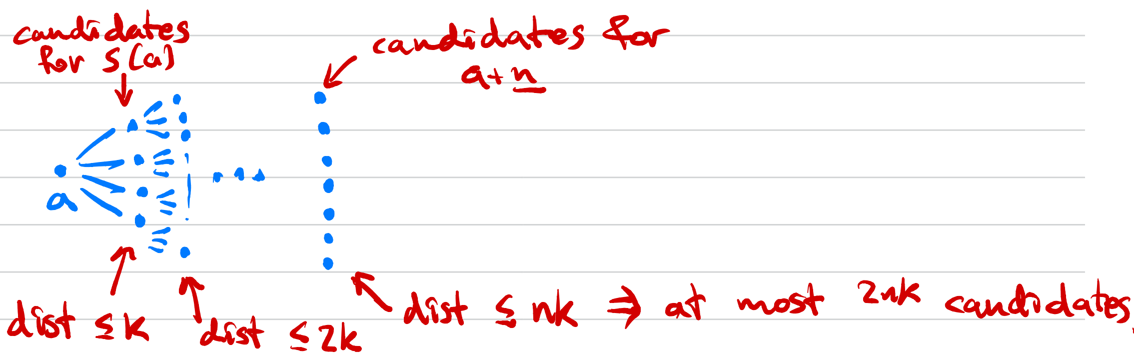
Recall

**Def** Given $M \models T$ and $a, b \in M$ the distance between a and b is the unique $k \in \mathbb{N}$ s.t. $a = b + \underline{k}$ or $b = a + \underline{k}$ or $\infty$ if no such $k$ exists

**Prop** If $\varphi(x,y)$ mut. alg. then there is some $k \in \mathbb{N}$ s.t. with only finitely many exceptions,

$$M \models \varphi(a,b) \Rightarrow \mathrm{dist}(a,b) \leq k$$

The point



The point Naive neighborhood guessing algorithm actually only generates $O(n)$ candidates for $a + \underline{n}$

There's still a problem $O(n)$ candidates for $a + \underline{n}$ does not imply $O(n^2)$ candidates for the entire sequence $a, a+\underline{1}, \ldots, a+\underline{n}$

But this problem is easy to fix (though technical) and the above point is really the key insight

## 5.3 Final guessing algorithm

**Prop** There is an algorithm which, given $n \in \mathbb{N}$, enumerates $O(n^2)$ guesses for the sequence $A(\underline{0}), A(\underline{1}), \ldots, A(\underline{n})$ at least one of which is correct

**Lemma** For every formula $\varphi(x)$ there is a number $k$ and an algorithm which, given $a \in M$ and the sequence $a-\underline{k}, \ldots, a+\underline{k}$, checks whether $M \models \varphi(a)$

Essentially follows from QE for T

**pf of Prop** $\varphi_A(x)$ = $\mathcal{L}'$-def. of $A$

Generate $O(n^2)$ guesses for $\underline{-k}, \ldots, \underline{0}, \ldots, \underline{n+k}$

For each guess, compute $A(\underline{0}), \ldots, A(\underline{n})$

## 6 Questions

1. Is there a natural theory with this property?
   I.e. a natural consistent, c.e. thy $T$ s.t. no thy def. equiv. to $T$ has a computable model?

2. Is there a natural consistent, c.e. theory $T$ which has no computable model but does not interpret any nontrivial fragment of arithmetic?

3. Is there any natural ctbl structure with no computable presentation?