Seetapun's Theorem and Kolmogorov Complexity

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Theorem (Seetapun). For every uncomputable X and set $A \subseteq \mathbb{N}$, either A or $\overline{A}$ $(= \mathbb{N} \setminus A)$ has an infinite subset which does not compute X.

Comments.

- Original motivation was reverse math of Ramsey's theorem
- First explicitly proved by Dzhafarov and Jockusch

Informally: You can't encode an infinite amount of information into all infinite subsets of both a set and its complement

Question. How much finite information can you encode?

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Meta question. How can we measure finite information?

Answer. Use Kolmogorov complexity.

Definition. For a string $\sigma \in 2^{<\omega}$ and set $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$, define

$$C(\sigma \mid \mathcal{X}) = \max_{B \in \mathcal{X}} C^B(\sigma).$$

Notation. For $A \subseteq \mathbb{N}$

- $[A]^\omega$ = set of infinite subsets of A.
- $\mathrm{Seet}(A) = [A]^\omega \cup [\overline{A}]^\omega$

Question, formal version. Given a string σ and set $A \subseteq \mathbb{N}$, how low can $C(\sigma \mid \mathrm{Seet}(A))$ be compared to $C(\sigma)$?

An Example

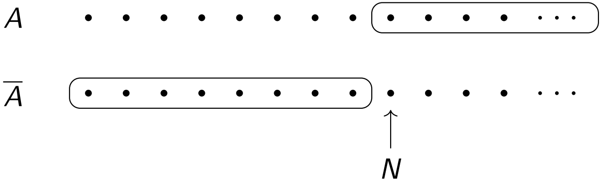
It is possible to encode "an arbitrary integer larger than N" (for any N).

Definition. For any string σ and number N, define

$$C(\sigma \mid \geq N) = \max_{n \geq N} C(\sigma \mid n).$$

Proposition. For any string σ and number N, there is some set $A \subseteq \mathbb{N}$ such that $C(\sigma \mid \text{Seet}(A)) \leq C(\sigma \mid \geq N) + O(1)$.

Proof.



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Theorem (Vereshchagin). For any string σ,

$$C^{0'}(\sigma) = \min_N C(\sigma \mid \geq N) \pm O(1).$$

So $C(\sigma \mid \mathrm{Seet}(A))$ can be as small as $C^{0'}(\sigma)$.

Question. Is there any way for all infinite subsets of both A and $\overline{A}$ to lower the complexity of σ below $C^{0'}(\sigma)$?

Answer. No.

The Main Theorem

Observation. $C(\sigma \mid \mathrm{Seet}(A))$ can be as small as $C^{0'}(\sigma)$.

Question. Is there any way for all infinite subsets of both A and $\overline{A}$ to lower the complexity of σ below $C^{0'}(\sigma)$?

Answer. No.

Theorem (Harrison-Trainor and L.). For all strings σ and sets $A \subseteq \mathbb{N}$,

$$C(\sigma \mid \mathrm{Seet}(A)) \geq C^{0'}(\sigma) - O(1).$$

Comment. Standard proofs of Seetapun's theorem don't seem to yield anything like this (at least not obviously).

How to Prove It*

*Sort of.

A much easier theorem. For all strings σ and sets $A \subseteq \mathbb{N}$,

$$C(\sigma \mid \mathrm{Seet}(A)) \geq C^X(\sigma) - O(\log |\sigma|)$$

where X is a complete Σ^1_2 set.

Proof strategy. Assume that for all $B \in \mathrm{Seet}(A)$, $C^B(\sigma) < k$ and show that $C^X(\sigma \mid k) \leq k + O(1)$.

Idea: Using X, enumerate a list of at most 2^k strings that "look like" σ i.e. a list of at most 2^k strings which includes σ

Key property of σ: There is some set A such that for all $B \in \mathrm{Seet}(A)$, $C^B(\sigma) < k$.

Claim 1. At most 2^k strings have this property.

Claim 2. X can enumerate the set of strings with this property.

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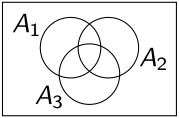
Proof. Suppose $\tau_1, \dots, \tau_n$ all have this property...
as witnessed by $A_1, \dots, A_n$.



Let B be a boolean combination of the A_i's which is infinite.

E.g. $B = A_1 \cap \overline{A_2} \cap \overline{A_3} \cap \dots \cap A_n$.

Then for each $i \leq n$, $C^B(\tau_i) < k$. Impossible if $n > 2^k$.

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Key property of σ: There is some set A such that for all $B \in \mathrm{Seet}(A)$, $C^B(\sigma) < k$.

Claim 2. X can enumerate the set of strings with this property.

Proof. The property is Σ^1_2.

A much easier theorem. For all strings σ and sets $A \subseteq \mathbb{N}$,

$$C(\sigma \mid \mathrm{Seet}(A)) \geq C^X(\sigma) - O(\log |\sigma|)$$

where X is a complete Σ^1_2 set.

Proof of easier theorem. Assume that for all $B \in \mathrm{Seet}(A)$, $C^B(\sigma) < k$. Identify a property of σ which is

- shared by at most 2^k other strings
- and which X can recognize.

Theorem (Harrison-Trainor and L.). For all strings σ and sets $A \subseteq \mathbb{N}$,

$$C(\sigma \mid \mathrm{Seet}(A)) \geq C^{0'}(\sigma) - O(1).$$

Proof idea. Identify a more complicated property of σ which is easier to compute.

Theorem (Harrison-Trainor and L.). For all strings σ and sets $A \subseteq \mathbb{N}$,

$$C(\sigma \mid \mathrm{Seet}(A)) \geq C^{0'}(\sigma) - O(1).$$

Proof sketch. Assume that for all $B \in \mathrm{Seet}(A)$, $C^B(\sigma) < k$.

Definition. A finite set of strings F is safe if there is some partition $A_1, \ldots, A_n$ of $\mathbb{N}$ such that for all $i \leq n$ and $s \subseteq A_i$ finite,

$$|s| > 1 \implies |\{\tau \mid C^s(\tau) < k\} \cup F| \leq 2^k.$$

i.e. we can safely assume that (all infinite subsets of) each A_i will give each $\tau \in F$ complexity less than k

Claim 1. No safe set has size larger than 2^k.

Claim 2. For any safe set F, $F \cup \{\sigma\}$ is also safe.

Therefore every maximal safe set contains σ.

Claim 3. The set of safe sets is $0'$-enumerable.

Therefore $0'$ can enumerate a maximal safe set.

Theorem (Harrison-Trainor and L.). For all strings σ and sets $A \subseteq \mathbb{N}$,

$$C(\sigma \mid \mathrm{Seet}(A)) \geq C^{0'}(\sigma) - O(1).$$

Proof sketch. Assume that for all $B \in \mathrm{Seet}(A)$, $C^B(\sigma) < k$.

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Claim 3. The set of safe sets is $0'$-enumerable.

The following $0'$-program enumerates a maximal safe set.

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Set F = ∅
While true:
    Search for τ such that F∪{τ} is safe
    Enumerate τ and set F = F∪{τ}
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Key point: A maximal safe set has size at most 2^k and contains σ

A Question

Theorem (Harrison-Trainor and L.). For all strings σ and sets $A \subseteq \mathbb{N}$,

$$C(\sigma \mid \mathrm{Seet}(A)) \geq C^{0'}(\sigma) - O(1).$$

In one sense, this theorem is sharp. But it doesn't seem to completely capture the following intuition.

Intuition. The only thing you can encode into all infinite subsets of both a set and its complement is "an arbitrary integer larger than N" for any single integer N.

Question. Fix a set $A \subseteq \mathbb{N}$. Is there a number N such that for all strings σ,

$$C(\sigma \mid \mathrm{Seet}(A)) \geq C(\sigma \mid \geq N) - O(1)?$$