Three theorems, three theorems, and three questions about Martin's conjecture JMM 2021

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What does the title mean?

Martin's conjecture is an ambitious attempt to classify the behavior of all definable functions on the Turing degrees.

I will discuss three old results on Martin's conjecture, three updates on those old results, and three questions suggested by those updates.

But first I will explain what Martin's conjecture is.

Where are the natural intermediate degrees?

It is easy to construct Turing degrees in-between 0 and $0'$. So why are all the undecidable problems that come up in mathematics at least as hard as the halting problem?

Martin's conjecture provides a partial explanation of this phenomenon.

The idea is that natural problems can be used to define operators on the Turing degrees.

Propaganda for Martin's conjecture

A "natural" undecidable problem A should be

- **Relativizable:** for each oracle $X \subseteq \mathbb{N}$, we have a version of the problem A relative to X, i.e. A defines an operator $X \mapsto A(X)$
- **Degree invariant:** equivalent oracles give equivalent versions of the problem, i.e. if $X \equiv_T Y$ then $A(X) \equiv_T A(Y)$

The point: A induces a function on the Turing degrees. We say that functions like this are Turing invariant. Martin's conjecture classifies such functions.

Possible Turing invariant functions

What Turing invariant functions can you think of?

Smart-aleck answer: Constant functions and the identity

An old favorite: The Turing jump

More: double jump, triple jump, ..., hyperjump, ..., sharps, ...

Is that everything (up to Turing equivalence)?

Possible Turing invariant functions

Turing invariant functions: constant functions, identity, iterates of the jump

Is that everything (up to Turing equivalence)? No!

Theorem (Kleene-Post): $\exists y\,(0 <_T y <_T 0')$

Relativized version: $\forall x\,\exists y\,(x <_T y <_T x')$

Example: For every Turing degree x, use choice to pick a y such that $x <_T y <_T x'$

Did we really need the axiom of choice?

Possible functions on the Turing degrees

Example 1: For every degree x, use choice to pick a y such that $x <_T y <_T x'$

Did we really need the axiom of choice? No!

Example 2: Fix a Turing degree z and define

$$f(x) = \begin{cases} 0 & \text{if } x \not\geq_T z \\ x' & \text{if } x \geq_T z. \end{cases}$$

But once you get above z, this is just the jump

Martin's conjecture, informally

Idea of Martin's conjecture: Exclude these types of examples

- Look at the behavior of functions "in the limit"
- Replace the axiom of choice with the axiom of determinacy (AD)

Martin's conjecture, super informal version: Under AD, every Turing invariant function is eventually equivalent to either a constant function, the identity or a transfinite iterate of the Turing jump

What does "in the limit" mean?

Definition: A cone of Turing degrees is a set of the form $\{x \mid x \geq_T y\}$ ("the cone above y")

If f and g are Turing invariant functions:

Definition: $f \equiv_M g$ if $f(x) \equiv_T g(x)$ on a cone

Definition: $f \leq_M g$ if $f(x) \leq_T g(x)$ on a cone

"f is constant on a cone" = f is equivalent to a constant function

"f is above the identity on a cone" = $f \geq_M \mathrm{id}$

What does "in the limit" mean?

The measure theory perspective

If A is a set of Turing degrees:

A has measure 1 if A contains a cone

A has measure 0 if A is disjoint from a cone

This forms a $\{0, 1\}$-valued measure on the Turing degrees, called "Martin measure"

$f \equiv_M g$ = "$f(x) \equiv_T g(x)$ for almost every x"

$f \leq_M g$ = "$f(x) \leq_T g(x)$ for almost every x"

The axiom of determinacy

Martin's conjecture replaces the axiom of choice with the axiom of determinacy. Why?

Philosophical reason 1: Proving Martin's conjecture under AD means that ZF cannot prove the existence of weird counterexamples

Philosophical reason 2: Limited forms of determinacy are provable for most reasonable classes of "definable functions." Borel determinacy is in ZF.

Practical reason: AD helps prove structural theorems, plays well with Martin measure

How to use determinacy in computability theory

Assuming the axiom of determinacy:

Fact: Every set of Turing degrees either contains a cone or is disjoint from a cone

Fact, restated: The Martin measure is an ultrafilter

Fact, restated again: If a set of Turing degrees is cofinal (for all x, there is some $y \geq_T x$ in the set) then it contains a cone

The first principle of using determinacy in computability: Describe what you want, show it holds cofinally, and let determinacy do the rest

How to use determinacy in computability theory

Example: Jump inversion via nuclear flyswatter

Theorem (jump inversion): Every large enough Turing degree is the jump of something

Formal version: There is some z such that for every $x \geq_T z$ there is some y such that $y' \equiv_T x$.

Proof: Let $A = \{x \mid \exists y\,(y' \equiv_T x)\}$. This set is cofinal because for each x, x' is above x and is in A. So by determinacy, A contains a cone.

This is a little silly because the Friedberg jump inversion theorem already says that this holds on the cone above 0'

Formal statement of Martin's conjecture

Martin's conjecture: Assuming the axiom of determinacy

(1) Every Turing invariant function is either constant on a cone or above the identity on a cone

(2) The (equivalence classes of) functions which are above the identity on a cone are well-ordered by $\leq_M$ and the successor in this well-order is given by the Turing jump

Some past results

Several special cases are known.

Uniformly invariant functions	Lachlan, Slaman, Steel
Regressive functions	Slaman, Steel
Part 2, order-preserving functions	Slaman, Steel

I will give an update on each of these results.

Uniformly invariant functions

Definition: If $x \equiv_T y$ via (i, j) means that $\Phi_i(x) = y$ and $\Phi_j(y) = x$

Definition: A Turing invariant function f is uniformly invariant if there is a function $u : \mathbb{N}^2 \to \mathbb{N}^2$ such that

$$x \equiv_T y \text{ via } (i, j) \implies f(x) \equiv_T f(y) \text{ via } u(i, j)$$

Theorem (Lachlan): If W is a uniformly invariant r.e. operator such that $W(x) \geq_T x$ for all x then either $W(x) \equiv_T x$ on a cone or $W(x) \equiv_T x'$ on a cone

Theorem (Steel, Slaman-Steel): Martin's conjecture holds for all uniformly invariant functions

The local perspective

Vittorio Bard has found a new way to understand some of these results.

Key idea: definition of uniformly invariant function still makes sense for a function defined on a single degree.

Notation: If x is a real, let $[x]_T$ denote the set of reals Turing equivalent to x

Definition: $f : [x]_T \to [y]_T$ is uniformly invariant if there is a function $u \colon \mathbb{N}^2 \to \mathbb{N}^2$ such that

$$x_0 \equiv_T x_1 \text{ via } (i, j) \implies f(x_0) \equiv_T f(x_1) \text{ via } u(i, j).$$

Bard's theorem

Definition: $f : [x]_T \to [y]_T$ is uniformly invariant if there is a function $u \colon \mathbb{N}^2 \to \mathbb{N}^2$ such that

$$x_0 \equiv_T x_1 \text{ via } (i, j) \implies f(x_0) \equiv_T f(x_1) \text{ via } u(i, j).$$

Idea: View $[x]_T$ together with action of Turing functionals as an algebraic structure. f and u together form a homomorphism

Theorem (Bard): If $f : [x]_T \to [y]_T$ is uniformly invariant then either f is constant or $x \leq_T y$

Key lemma: Can always assume u is computable

Bard's theorem

Theorem (Bard): If $f : [x]_T \to [y]_T$ is uniformly invariant then either f is constant or $x \leq_T y$

By determinacy, this theorem easily implies part 1 of Martin's conjecture for uniformly invariant functions.

Very sketchy proof: Either $f(x) \geq_T x$ on a cone or $f(x) \ngeq_T x$ on a cone. In the latter case, Bard's theorem implies f is actually a function from Turing degrees to reals. Use determinacy to fix each bit of the output.

Obvious question: Does part 2 also arise locally?

Part 2 locally?

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Theorem (Bard-L.): Suppose $x \geq_T 0'$ and W is an r.e. operator which is uniformly invariant on $[x]_T$. Then either $W(x) \equiv_T x'$ or $W(x) \equiv_T x$ or W is constant on $[x]_T$.

This is the local version of Lachlan's result. What about the local version for the full part 2?

Question: Suppose $f : [x]_T \to [y]_T$ is uniformly invariant. Must it be the case that either f is continuous or $f(x) \geq_T x'$?

Regressive functions

Definition: A Turing invariant function f is called a regressive function on the Turing degrees if for all x, $f(x) \leq_T x$

Theorem (Slaman-Steel): If f is a regressive function on the Turing degrees then either f is constant on a cone or $f(x) \equiv_T x$ on a cone.

Idea of Slaman and Steel's proof: Condition on f means we can basically assume it's continuous. Combine this with a clever coding argument.

Obvious question: Does this approach still work when you can't assume f is continuous?

Is continuity necessary?

Idea of Slaman and Steel's proof: Condition on f means we can basically assume it's continuous. Combine this with a clever coding argument.

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Question asked by Slaman-Steel: Does the analogous theorem hold on the hyperarithmetic degrees?

The point is that a regressive function on the hyperarithmetic degrees can only be assumed to be Borel, not continuous.

Continuity is not necessary (but it is necessary)

Question asked by Slaman-Steel: Does the analogous theorem hold on the hyperarithmetic degrees?

Theorem (L.): Suppose f is a hyp-invariant function such that $f(x) \leq_H x$ for all x. Then either f is constant on a cone of hyperarithmetic degrees or $f(x) \equiv_H x$ on a cone of hyperarithmetic degrees.

Key idea: We can't assume f is continuous, but we can replace f with a hyp-equivalent continuous function. We can then use a coding argument to finish.

Replacing with a continuous function

Key idea: We can't assume f is continuous, but we can replace f with a hyp-equivalent continuous function.

Very sketchy proof: Want to find g such that $g(x) \leq_T x$ and $g(x) \equiv_H f(x)$ on a cone. It's enough to show that the following set contains a cone

$$A = \{x \mid \exists y\, (y \leq_T x \text{ and } y \equiv_H f(x))\}.$$

By determinacy, enough to show A is cofinal. Start with any z. Since $f(z) \leq_H z$, there is some $x \equiv_H z$ such that $f(z) \leq_T x$. Then $x \oplus z$ is in A, as witnessed by $f(z)$.

Replacing with a continuous function

Key idea: We can't assume f is continuous, but we can replace f with a hyp-equivalent continuous function.

This trick works for many degree structures. But surprisingly, it is harder to adapt the coding argument.

Question: Does Martin's conjecture hold for regressive functions on the arithmetic degrees?

Note that there is a known counterexample to Martin's conjecture on the arithmetic degrees.

Order preserving functions

Definition: A Turing invariant function f is order preserving if for all x and y

$$x \geq_T y \implies f(x) \geq_T f(y).$$

Theorem (Slaman-Steel): Part 2 of Martin's conjecture holds for all Borel, order preserving functions which are above the identity.

Obvious question: What about part 1?

Order preserving functions

Obvious question: What about part 1?

An interesting idea from Takayuki Kihara: You can use the Solecki dichotomy to show that if f is an order preserving function then either f is constant on a cone or there is some real a such that on a cone, $f(x) \oplus a \geq_T x'$.

Solecki dichotomy: Roughly says that (under AD) for any function f on the reals, either f is a countable union of partial continuous functions or f "embeds" the Turing jump

A different approach

Obvious question: What about part 1?

Theorem (L.-Siskind): Part 1 of Martin's conjecture holds for all order preserving functions.

The proof can be broken down into three parts:

(1) Define a new class of functions—"measure preserving functions"

(2) Show that every order preserving function is either constant on a cone or measure preserving

(3) Show that part 1 of Martin's conjecture holds for all measure preserving functions

What is a measure preserving function?

Definition: A Turing invariant function f is measure preserving if for all x, there is some y such that

$$z \geq_T y \implies f(z) \geq_T x.$$

"f goes to infinity in the limit"

Theorem (L.-Siskind): If f is an order preserving function then either f is constant on a cone or f is measure preserving.

Proved using a new basis theorem for perfect sets.

Interesting point: This theorem is exactly what's needed to finish Kihara's proof

I think that understanding this better may prove important.

A surprise

Equivalent Definition of measure preserving: A Turing invariant function f is measure preserving if the function it induces on the Turing degrees preserves the Martin measure (in the sense of ergodic theory).

Theorem (L.-Siskind): Part 1 of Martin's conjecture holds for all measure preserving functions

This theorem connects Martin's conjecture to a statement about the Rudin-Keisler order on ultrafilters on the Turing degrees!

Some measure theory background

If μ is a measure on X and $f : X \to Y$ is a function

Definition: The pushforward of μ by f is the measure given by $f_*\mu(A) = \mu(f^{-1}(A))$

Definition: f is measure preserving if $X = Y$ and $f_*\mu = \mu$

If μ and ν are ultrafilters on X

Definition (Rudin-Keisler order): $\mu \leq_{RK} \nu$ if there is some $f : X \to X$ such that $f_*\nu = \mu$

Principal ultrafilters are $\leq_{RK}$-minimal. If X is an ordinal then the normal ultrafilters are $\leq_{RK}$-minimal above the principal ultrafilters

Martin's conjecture and the Rudin-Keisler order

Equivalent definition, restated: A function f on the Turing degrees is measure preserving if and only if

$$f_*(\text{Martin measure}) = \text{Martin measure}$$

Corollary: Part 1 of Martin's conjecture holds iff for all nonprincipal ultrafilters μ on the Turing degrees

- μ is not strictly below Martin measure in the Rudin-Keisler order
- and if μ is Rudin-Keisler equivalent to Martin measure then μ is Martin measure

Can we at least prove one of the two bullet points?

Martin's conjecture and the Rudin-Keisler order

Question: Is there any ultrafilter on the Turing degrees which is strictly below Martin measure in the Rudin-Keisler order?

One way to give a positive answer is to prove the following statement, which was asked by Andrew Marks:

(Assuming AD) Every function f on the reals is either constant or injective on some pointed perfect tree.

Conclusions

Martin's conjecture is an old conjecture in computability theory.

Much of what's known about it was proved in the 1970s and 1980s.

But recently several promising new directions have been discovered.

What's next for Martin's conjecture?