Martin's conjecture above the hyperjump

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Question. What are the "natural" functions on the Turing degrees?

Answer. The Turing jump and its transfinite iterates.

constant functions	identity function	Turing jump	double jump	...	omega jump	...	hyper-jump
$x \mapsto c$	$x \mapsto x$	$x \mapsto x'$	$x \mapsto x''$	...	$x \mapsto x^{(\omega)}$	...	$x \mapsto \mathcal{O}^x$

Goal of this talk. Discuss work on formalizing intuition about natural functions on the Turing degrees using the framework of Martin's conjecture, with a focus on functions above the hyperjump.

Notation. $\mathcal{D}_T$ = Turing degrees, $Id(x)$ = identity function on $\mathcal{D}_T$, $J(x)$ = Turing jump, $H(x) = \mathcal{O}^x$ = hyperjump

Martin’s conjecture

Two examples of "weird" functions on the Turing degrees.

Example 1. For each x, use choice to pick y such that $x <_T y <_T x'$ and set $f(x) = y$.

Example 2. For each x, define

$$g(x) = \begin{cases} x & \text{if } x \not\geq_T 0' \\ x' & \text{if } x \geq_T 0'. \end{cases}$$

Idea of Martin's conjecture: Rule out such pathological functions by

- Looking at the behavior of functions "in the limit"
- Replacing the Axiom of Choice with the Axiom of Determinacy (AD)

Martin's conjecture, informally. Under AD, every function $f : \mathcal{D}_T \to \mathcal{D}_T$ is either eventually constant, eventually the identity or eventually a transfinite iterate of the Turing jump

Idea of Martin's conjecture: Rule out such pathological functions by

- Look at the behavior of functions "in the limit"
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Definition. A cone of Turing degrees is a set of the form $\{x \mid x \geq_T y\}$

Definition. Suppose $f, g \colon \mathcal{D}_T \to \mathcal{D}_T$.

- $f \equiv_M g$ if $f(x) \equiv_T g(x)$ on a cone
- $f \leq_M g$ if $f(x) \leq_T g(x)$ on a cone.

Martin's conjecture classifies functions on the Turing degrees up to $\equiv_M$ equivalence

Idea of Martin's conjecture: Rule out such pathological functions by

- Look at the behavior of functions "in the limit"
- Replace the Axiom of Choice with the Axiom of Determinacy (AD)

Axiom of Determinacy: Contradicts the Axiom of Choice and consistent relative to large cardinals

Theorem (Martin). Under AD, every set of Turing degrees either contains a cone or is disjoint from a cone.

Definition. A set $A \subseteq \mathcal{D}_T$ is cofinal if for all x, there is some $y \geq_T x$ such that $y \in A$.

Theorem, restated. Under AD, if a set of Turing degrees is cofinal then it contains a cone.

Note. There are a number of variations of AD, e.g. AD^+, $\mathrm{AD}_{\mathbb{R}}$, $\mathrm{AD} + V = L(\mathbb{R})$. We will conflate these for the purposes of this talk.

Theorem (Martin). Under AD, if a set of Turing degrees is cofinal then it contains a cone.

Example: jump inversion via nuclear flyswatter. For every x on some cone, there is some y such that $y' \equiv_T x$.

Proof. By AD, it is enough to show this is true cofinally. Fix a and we will show it holds above a.

Let $x = a'$. Then for some y (namely a itself), $y' \equiv_T x$. Since $x \geq_T a$ we are done.

Martin's conjecture. Assuming AD,

(1) Every function $f\colon \mathcal{D}_T \to \mathcal{D}_T$ is either constant on a cone or above the identity on a cone (i.e. $f \geq_M Id$).

(2) The $\equiv_M$-equivalence classes of functions which are above the identity on a cone are well-ordered by $\leq_M$ and the successor in this well-order is given by the Turing jump (i.e. the successor of f is $J \circ f$).

In a sense, the conjecture implies that every function above the identity is equivalent to a transfinite iterate of the jump, but it does not give us a concrete description of these functions.

Order-preserving functions

Definition. A function $f\colon \mathcal{D}_T \to \mathcal{D}_T$ is order-preserving if for all x, y

$$x \geq_T y \implies f(x) \geq_T f(y).$$

Theorem (L.-Siskind). Part 1 of Martin's conjecture holds for all order-preserving functions.

Theorem (Slaman-Steel). Part 2 of Martin's conjecture holds for all order-preserving functions $f \leq_M H$.

Actually, Slaman and Steel prove more: an explicit description of all order-preserving functions $f\colon \mathcal{D}_T \to \mathcal{D}_T$ such that $Id \leq_M f \leq_M H$.

Actually, Slaman and Steel prove more: an explicit description of all order-preserving functions $f\colon \mathcal{D}_T \to \mathcal{D}_T$ such that $Id \leq_M f \leq_M H$.

Recall. For any $x \in \mathcal{D}_T$, ω_1^x denotes the least ordinal with no presentation computable from x. If $\alpha < \omega_1^x$ then $x^{(\alpha)}$, the α^{th} jump of x, is well-defined and $x^{(\alpha)} <_T H(x)$.

Observation. For any $\alpha < \omega_1$, $J_\alpha(x) = x^{(\alpha)}$ is well-defined on a cone and $J_\alpha <_M H$.

Theorem (Slaman-Steel). For any order-preserving $f\colon \mathcal{D}_T \to \mathcal{D}_T$, if $Id \leq_M f \leq_M H$ then either

(1) there is some $\alpha < \omega_1$ such that $f \equiv_M J_\alpha$

(2) or $f \equiv_M H$.

Question 1. Can Slaman and Steel's results be extended to all order-preserving functions?

Question 2. Is there a concrete description of the functions above the hyperjump?

Functions above the hyperjump

Question. What are the "natural" functions between the hyperjump and the double hyperjump?

Some obvious examples. $H(x), J(H(x)), J_2(H(x)), \ldots, J_\alpha(H(x)), \ldots$

Question. Are there any others?

Answer. Yes! For example, define $f(x) = H(x)^{(\omega_1^x)}$.

Note that f is well-defined: for any x, $\omega_1^x < \omega_1^{H(x)} = \omega_2^x$.

A more general example. Let $\alpha\colon \mathcal{D}_T \to \omega_1$ be any function such that for all x, $\alpha(x) < \omega_2^x$. Define $f(x) = H(x)^{(\alpha(x))}$.

$f(x)$ is well-defined and below $H(H(x))$ on a cone.

A more general example. $f(x) = H(x)^{(\alpha(x))}$ for any function $\alpha\colon \mathcal{D}_T \to \omega_1$ such that $\alpha(x) < \omega_2^x$.

The class of functions $\mathcal{D}_T \to \omega_1$ is nicely behaved under AD.

Definition. For any functions $\alpha, \beta\colon \mathcal{D}_T \to \omega_1$, define $\alpha \leq_M \beta$ to mean $\alpha(x) \leq \beta(x)$ on a cone.

Proposition. The collection of functions $\mathcal{D}_T \to \omega_1$ is pre-well-ordered by $\leq_M$.

Corollary. The collection of functions

$$\{H(x)^{(\alpha(x))} \mid \alpha\colon \mathcal{D}_T \to \omega_1 \text{ and for all } x,\ \alpha(x) < \omega_2^x\}$$

is well-ordered by $\leq_M$.

Theorem (L.-Siskind). For any order-preserving $f\colon \mathcal{D}_T \to \mathcal{D}_T$, if $H \leq_M f <_M H \circ H$ then for some function $\alpha\colon \mathcal{D}_T \to \omega_1$, $f \equiv_M H(x)^{(\alpha(x))}$.

The proof of the Slaman-Steel theorem

Theorem (Slaman-Steel). For any order-preserving $f : \mathcal{D}_T \to \mathcal{D}_T$, if $Id \leq_M f <_M H$ then there is some $\alpha < \omega_1$ such that $f \equiv_M J_\alpha$.

Theorem (Posner-Robinson). For any $x >_T 0$, there is some g such that $x \oplus g \geq_T g'$.

If x is not computable, it looks like the jump relative to some g.

Generalized Posner-Robinson Theorem (Jockusch-Shore). Fix x and $\alpha < \omega_1^{CK}$ and suppose that for all $\beta < \alpha$, $0^{(\beta)} <_T x$. Then there is some g such that $x \oplus g \geq_T g^{(\alpha)}$.

If x is strictly above $0^{(\beta)}$ for each $\beta < \alpha$ then x looks like the α-jump relative to some g.

Relativized version. Fix x, y and $\alpha < \omega_1^x$ and suppose that for all $\beta < \alpha$, $x^{(\beta)} <_T y$. Then there is some $g \geq_T x$ such that $y \oplus g \geq_T g^{(\alpha)}$.

Theorem (Slaman-Steel). For any order-preserving $f\colon \mathcal{D}_T \to \mathcal{D}_T$, if $Id \leq_M f <_M H$ then there is some $\alpha < \omega_1$ such that $f \equiv_M J_\alpha$.

Proof. Let α be the least ordinal such that for all x on a cone, $f(x)$ is not strictly above $x^{(\alpha)}$. We will show $f \equiv_M J_\alpha$.

Enough to show $f(x) \geq_T x^{(\alpha)}$ cofinally. So fix x and we will show this happens somewhere above x.

For every $\beta < \alpha$, $x^{(\beta)} <_T f(x)$. So by the generalized Posner-Robinson theorem, there is some $g \geq_T x$ such that $f(x) \oplus g \geq_T g^{(\alpha)}$.

Now calculate:

$$\begin{aligned}
f(g) &\geq_T f(x) \oplus f(g) && (f \text{ is order-preserving and } x \leq_T g)\\
&\geq_T f(x) \oplus g && (f \geq_M Id)\\
&\geq_T g^{(\alpha)} && (\text{by choice of } g).
\end{aligned}$$

Theorem (L.-Siskind). For any order-preserving $f\colon \mathcal{D}_T \to \mathcal{D}_T$, if $H \leq_M f <_M H \circ H$ then for some $\alpha\colon \mathcal{D}_T \to \omega_1$, $f \equiv_M H(x)^{\alpha(x)}$.

Proof attempt. Let ~~α be the least ordinal such that for all x on a cone, $f(x)$ is not strictly above $x^{(\alpha)}$~~ $\alpha(x) =$ least ordinal such that $f(x)$ is not strictly above $H(x)^{(\alpha(x))}$. We will show ~~$f \equiv_M J_\alpha$~~ $f(x) \equiv_M H(x)^{(\alpha(x))}$.

Enough to show ~~$f(x) \geq_T x^{(\alpha)}$~~ $f(x) \geq_T H(x)^{(\alpha(x))}$ cofinally. So fix x and we will show this happens somewhere above x.

For every ~~$\beta < \alpha$, $x^{(\beta)} <_T f(x)$~~ $\beta < \alpha(x)$, $H(x)^{(\beta)} <_T f(x)$. So by the generalized Posner-Robinson theorem, there is some $g \geq_T x$ such that ~~$f(x) \oplus g \geq_T g^{(\alpha)}$~~ $f(x) \oplus g \geq_T H(g)^{(\alpha(x))}$.

Now calculate:

$$\begin{aligned} f(g) &\geq_T f(x) \oplus f(g) && (f \text{ is order-preserving and } x \leq_T g) \\ &\geq_T f(x) \oplus g && (f \geq_M Id) \\ &\geq_T H(g)^{(\alpha(x))} && (\text{by choice of } g). \end{aligned}$$

Two problems: first, need an appropriate Posner-Robinson theorem. Second, need $f(g) \geq_T H(g)^{(\alpha(g))}$ not $H(g)^{(\alpha(x))}$.

Fixing the proof

Theorem (L.-Siskind). For any order-preserving $f\colon \mathcal{D}_T \to \mathcal{D}_T$, if $H \leq_M f <_M H \circ H$ then for some $\alpha\colon \mathcal{D}_T \to \omega_1$, $f \equiv_M H(x)^{\alpha(x)}$.

Problems.

(1) Need an appropriate version of the Posner-Robinson theorem.

(2) Need to know that the real g witnessing the Posner-Robinson theorem preserves α, i.e. $\alpha(g) = \alpha(x)$.

Problem 1. Need an appropriate version of Posner-Robinson theorem.

Solution 1. The necessary version of Posner-Robinson can be proved using Jockusch-Shore + hyperjump inversion.

Theorem. Fix x and $\alpha < \omega_2^{CK}$. Suppose that for all $\beta < \alpha$, $\mathcal{O}^{(\beta)} <_T x$. Then for some g, $x \oplus g \geq_T H(g)^{(\alpha)}$.

Proof. By the generalized Posner-Robinson theorem relative to $\mathcal{O}$, there is some $h \geq_T \mathcal{O}$ such that $x \oplus h \geq_T h^{(\alpha)}$.

Since $h \geq_T \mathcal{O}$, we can apply hyperjump inversion to get g such that

$$H(g) \equiv_T g \oplus \mathcal{O} \equiv_T h.$$

Now calculate

$$\begin{aligned} x \oplus g &\geq_T x \oplus h && (x \geq_T \mathcal{O} \text{ and } g \oplus \mathcal{O} \geq_T h) \\ &\geq_T h^{(\alpha)} && (\text{by choice of } h) \\ &\geq_T H(g)^{(\alpha)} && (\text{since } H(g) \equiv_T h). \end{aligned}$$

Problem 2. Need the real g witnessing Posner-Robinson to preserve $\alpha\colon \mathcal{D}_T \to \omega_1$.

Solution 2. Classification of functions $\mathcal{D}_T \to \omega_1$ below $x \mapsto \omega_2^x$.

Theorem (Siskind). If $\alpha\colon \mathcal{D}_T \to \omega_1$ is strictly below $x \mapsto \omega_2^x$ then there is some $\sigma\colon \omega_1 \to \omega_1$ such that on a cone, $\alpha(x) = \sigma(\omega_1^x)$.

The proof works by comparing the ultrapower by the cone measure on $\mathcal{D}_T$ to the iterated ultrapower by the club filter.

Upshot. It is enough to show that g preserves ω_1^{CK}.

But this follows from our proof of the existence of g.

Recall that g was found by applying hyperjump inversion to some $h \geq_T \mathcal{O}$. And it is well-known that

$$H(g) \equiv_T g \oplus \mathcal{O} \implies \omega_1^g = \omega_1^{CK}.$$

Past the double hyperjump

These ideas work well past the double hyperjump.

In particular: there is some long initial segment of the $\leq_M$ order where every function has the form $x \mapsto H^{\alpha(x)}(x)^{(\beta(x))}$ for some functions $\alpha, \beta \colon \mathcal{D}_T \to \omega_1$.

The three ingredients.

(1) An appropriate version of the Posner-Robinson theorem
(2) A classification of functions $\mathcal{D}_T \to \omega_1$
(3) A proof that the real witnessing the Posner-Robinson theorem preserves appropriate ordinal-valued functions.

Ingredient 1 follows relatively easily from known proofs of the Posner-Robinson theorem for the hyperjump (due to Slaman and written up by Jananthan and Simpson)

Ingredient 2: classification of functions $\mathcal{D}_T \to \omega_1$

Notation. For any $\alpha \colon \mathcal{D}_T \to \omega_1$, let $|\alpha|$ denote the rank of α in the $\leq_M$ well-order.

Technical definition. Let C be the set of uniform indiscernibles, i.e. ordinals α such that for all $x \in 2^\omega$, α is a Silver indiscernible for $L[x]$.

Theorem (Siskind). For every function $\alpha \colon \mathcal{D}_T \to \omega_1$, there is some $\sigma \colon \omega_1^n \to \omega_1$ and functions $\beta_1, \ldots, \beta_n \colon \mathcal{D}_T \to \omega_1$ such that

$$\alpha(x) = \sigma(\beta_1(x), \ldots, \beta_n(x))$$

and for each i, $|\beta_i| \in C$.

Upshot. When proving instances of Martin's conjecture for order-preserving functions using the strategy explained earlier, it suffices to preserve ordinal valued functions whose ranks are in C.

Ingredient 3: Preserving ordinal valued functions.

Definition. An ordinal $\alpha < \omega_1$ is admissible if for some x, $\alpha = \omega_1^x$ and admissible relative to x if for some $y \geq_T x$, $\alpha = \omega_1^y$.

Theorem (Steel). For some long initial segment of $\leq_M$, if $\alpha\colon \mathcal{D}_T \to \omega_1$ has rank in C then for all x on a cone, $\alpha(x)$ is either admissible relative to x or the limit of ordinals which are admissible relative to x.

Thus it makes sense to focus on proving that the Posner-Robinson theorem has witnesses which preserve admissibility.

Preserving admissibility

In many cases, it is possible to prove versions of the Posner-Robinson theorem while preserving admissibility of many ordinals.

Kumabe-Slaman forcing. Conditions consist of a finite set $\Phi \subseteq \omega$ and a finite set $F \subseteq 2^\omega$.

Key point. The generic is formed using only the finite subsets of ω and not the finite subsets of 2^ω.

Theorem. Generics for Kumabe-Slaman forcing preserve admissibility.

Key lemma (Slaman/Jananthan-Simpson). Suppose M is a model of ZFC, (Φ, F) is a Kumabe-Slaman condition in M, D is a dense set for Kumabe-Slaman forcing in M and G is any finite set of reals. Then in M there is some $(\Phi', F') \leq (\Phi, F)$ meeting D such that (Φ', F') is compatible with $(\Phi, F \cup G)$.

The point. If M is a model of ZFC then any Kumabe-Slaman generic is also a Kumabe-Slaman generic over M, even though M may not actually contain all Kumabe-Slaman conditions.

How far does this work?

It is not clear how far we can go using these ideas. But here is a natural guess.

Definition. The least recursively inaccessible is the least ordinal less than ω_1 which is both admissible and a limit of admissibles.

This ordinal is extremely large compared to ω_1^{CK}.

Notation. Let $\kappa\colon \mathcal{D}_T \to \omega_1$ denote the function $\kappa(x) =$ least recursively inaccessible relative to x.

Fact. Iterating the hyperjump is well-defined below the least recursively inaccessible.

Natural guess. Fix functions $\alpha, \beta\colon \mathcal{D}_T \to \omega_1$ such that for all x

- $\alpha(x) < \kappa(x)$
- $\beta(x) < \omega_{\alpha(x)}^x$

and define $f(x) = H^{\alpha(x)}(x)^{(\beta(x))}$.

The collection of such functions is well-ordered by $\leq_M$. It seems natural to guess that this collection forms an initial segment of $\leq_M$.

The main obstacle to extending our analysis through the first recursively inaccessible:

Definition. A function $\alpha \colon \mathcal{D}_T \to \omega_1$ is uniformly singular if there are functions $\beta, \{\gamma_i\}_{i<\omega_1} \colon \mathcal{D}_T \to \omega_1$ all strictly $\leq_M$-below α such that for all x,

$$\alpha(x) = \sup_{i<\beta(x)} \gamma_i(x).$$

Example. Define $\alpha(x) = \omega_1^x \cdot 2$. Then α is uniformly singular, as witnessed by $\beta(x) = \omega_1^x$ and $\gamma_i(x) = \omega_1^x + i$.

Fact. The function $x \mapsto \omega_1^x$ is not uniformly singular.

Question. Suppose $\alpha \leq_M \kappa$ and $\alpha(x)$ is not admissible on a cone. Is α always uniformly singular?