Part 1 of Martin's Conjecture and Measure Preserving Functions

Patrick Lutz (joint work with Benny Siskind)

University of California, Berkeley

TL;DR

Martin's conjecture: classifies the "in the limit" behavior of definable functions on the Turing degrees.

Our result: We identify a new natural class of functions on the Turing degrees and prove part of Martin's conjecture for this class.

This also allows us to extend some old results on Martin's conjecture.

Possible functions on the Turing degrees

What functions on the Turing degrees can you think of?

Smart-aleck answer: Constant functions and the identity

An old favorite: The Turing jump

More: double jump, triple jump, ..., hyperjump, ..., sharps, ...

Is that everything?

Possible functions on the Turing degrees

Functions on the Turing degrees: constant functions, identity, iterates of the jump

Is that everything? No!

Theorem (Kleene-Post): $\exists y\,(0 <_T y <_T 0')$

Relativized version: $\forall x\,\exists y\,(x <_T y <_T x')$

Example: For every x, use choice to pick a y such that $x <_T y <_T x'$

Did we really need the axiom of choice?

Possible functions on the Turing degrees

Example 1: For every x, use choice to pick a y such that $x <_T y <_T x'$

Did we really need the axiom of choice? No!

Example 2: Fix a Turing degree z and define

$$f(x) = \begin{cases} 0 & \text{if } x \not\geq_T z \\ x' & \text{if } x \geq_T z. \end{cases}$$

But this example is pretty lame

Martin's conjecture, informally

Idea of Martin's conjecture: Exclude these types of examples

- Look at the behavior of functions "in the limit"
- Replace the axiom of choice with the axiom of determinacy (AD)

Martin's conjecture, super informal version: Under AD, every function on the Turing degrees is either eventually constant, eventually the identity or eventually a transfinite iterate of the Turing jump

What does "in the limit" mean?

Definition: A cone of Turing degrees is a set of the form $\{x \mid x \geq_T y\}$ ("the cone above y")

If f and g are functions on the Turing degrees:

Definition: $f \equiv_M g$ if $f(x) \equiv_T g(x)$ on a cone

Definition: $f \leq_M g$ if $f(x) \leq_T g(x)$ on a cone

"f is constant on a cone" $=$ f is equivalent to a constant function

"f is above the identity on a cone" $=$ $f \geq_M \mathrm{id}$

What does “in the limit” mean?

The measure theory perspective

If A is a set of Turing degrees:

A has measure 1 if A contains a cone

A has measure 0 if A is disjoint from a cone

This forms a $\{0, 1\}$-valued measure on the Turing degrees, called “Martin measure”

$f \equiv_M g$ = “$f(x) \equiv_T g(x)$ for almost every x”

$f \leq_M g$ = “$f(x) \leq_T g(x)$ for almost every x”

The axiom of determinacy

Martin's conjecture replaces the axiom of choice with the axiom of determinacy. Why?

Philosophical reason 1: Proving Martin's conjecture under AD means that ZF cannot prove the existence of weird counterexamples

Philosophical reason 2: Limited forms of determinacy are provable for most reasonable classes of "definable functions"

Practical reason: AD helps prove structural theorems, plays well with Martin measure

How to use determinacy in computability theory

Assuming the axiom of determinacy:

Fact: Every set of Turing degrees either contains a cone or is disjoint from a cone

Fact, restated: The Martin measure is an ultrafilter

Fact, restated again: If a set of Turing degrees is cofinal (for all x, there is some $y \geq_T x$ in the set) then it contains a cone

The first principle of using determinacy in computability: Describe what you want, show it holds cofinally, and let determinacy do the rest

How to use determinacy in computability theory

Example: Jump inversion via nuclear flyswatter

Theorem (jump inversion): Every large enough Turing degree is the jump of something

Formal version: There is some z such that for every $x \geq_T z$ there is some y such that $y' \equiv_T x$.

Proof: Let $A = \{x \mid \exists y\, (y' \equiv_T x)\}$. This set is cofinal because for each x, x' is above x and is in A. So by determinacy, A contains a cone.

This is a little silly because the Friedberg jump inversion theorem already says that this holds on the cone above 0'

Formal statement of Martin's conjecture

Martin's conjecture: Assuming the axiom of determinacy

(1) Every function on the Turing degrees is either constant on a cone or above the identity on a cone

(2) The (equivalence classes of) functions which are above the identity on a cone are well-ordered by $\leq_M$ and the successor in this well-order is given by the Turing jump

Disclaimer: Martin's conjecture is usually stated in terms of Turing-invariant functions on 2^ω. Assuming $AD_{\mathbb{R}}$, these two definitions are equivalent.

Results on Martin's conjecture

Some special cases of Martin's conjecture are known

Definition: A function f on the Turing degrees is order preserving if for all x and y,

$$x \geq_T y \implies f(x) \geq_T f(y)$$

Definition: A function f on the Turing degrees is measure preserving if for all x there is y such that

$$z \geq_T y \implies f(z) \geq_T x$$

measure preserving $=$ "goes to infinity in the limit"

Results on Martin's conjecture

Some special cases of Martin's conjecture are known

Theorem (Slaman and Steel): Part 1 holds for functions which are below the identity on a cone

Theorem (Slaman and Steel): Part 2 holds for order-preserving Borel functions

Theorem (L. and Siskind): Part 1 holds for measure preserving functions*

Theorem (L. and Siskind): Part 1 holds for order preserving functions (rules out "sideways" functions)

*Requires either $AD_{\mathbb{R}}$ or AD^{+}

Measure preserving functions

Part 1 of Martin's conjecture holds for measure preserving functions. Why should I care?

Measure preserving functions are natural:

They are exactly the functions which are above every constant function in the Martin order

They are exactly the functions which preserve Martin measure in the sense of ergodic theory

Corollary: Part 1 of Martin's conjecture is equivalent to a statement about the Rudin-Keisler order

Measure preserving functions

Part 1 of Martin's conjecture holds for measure preserving functions. Why should I care?

Yields other results about Martin's conjecture:

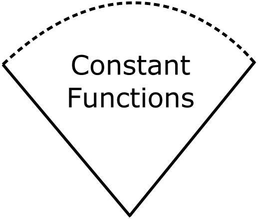
Order preserving functions are either constant on a cone or measure preserving

Corollary 1: Part 1 of Martin's conjecture holds for order preserving functions

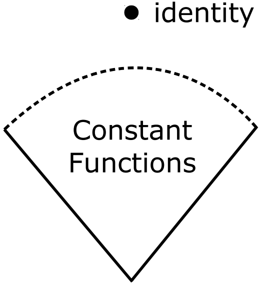
Corollary 2: The Turing degrees are not a universal locally countable Borel partial order

If f and g are above the identity and for all x and y, $f(x) \equiv_T f(y)$ implies $g(x) \equiv_T g(y)$ then $f \leq_M g$

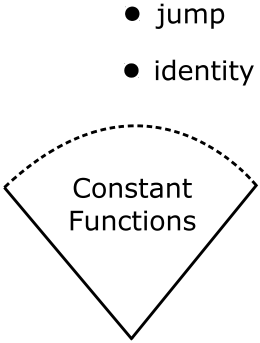
Picture of functions on the Turing degrees



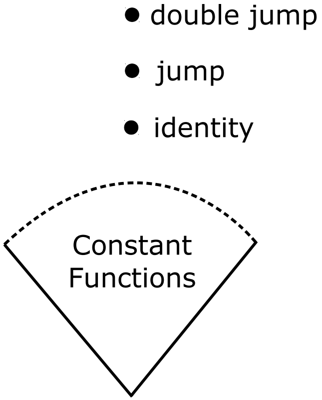
Picture of functions on the Turing degrees



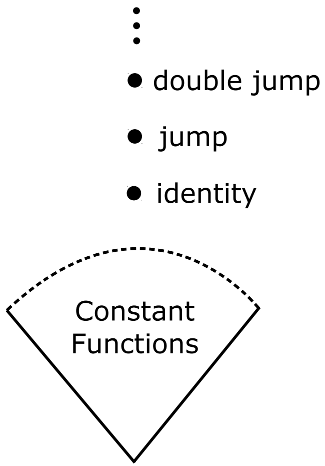
Picture of functions on the Turing degrees



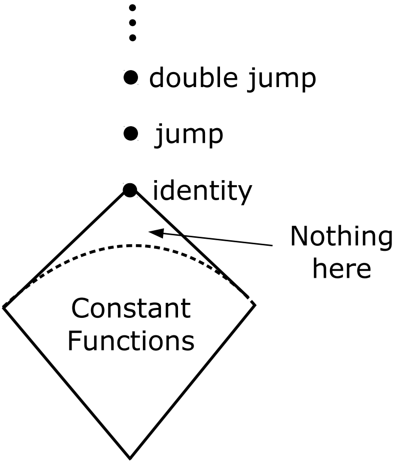
Picture of functions on the Turing degrees



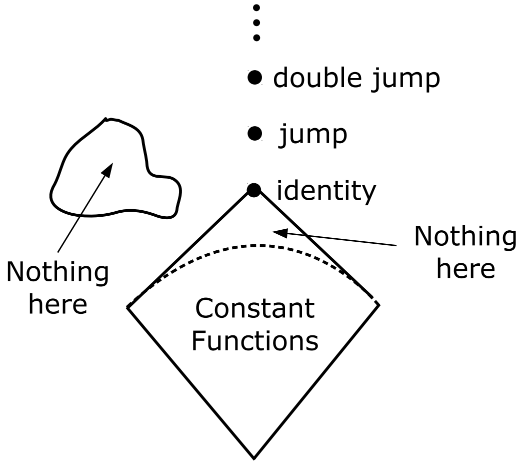
Picture of functions on the Turing degrees



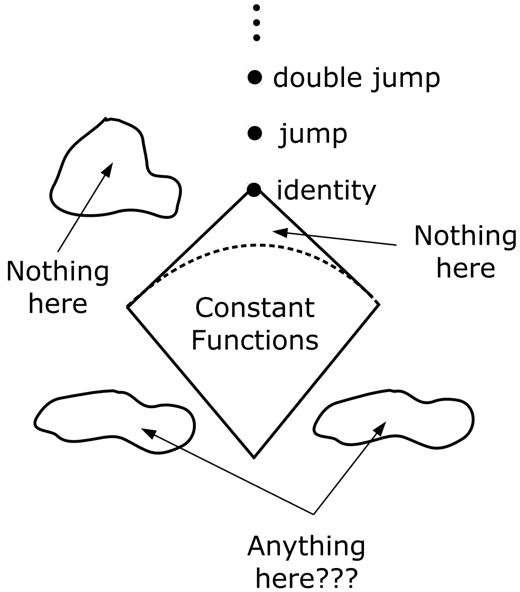
Picture of functions on the Turing degrees



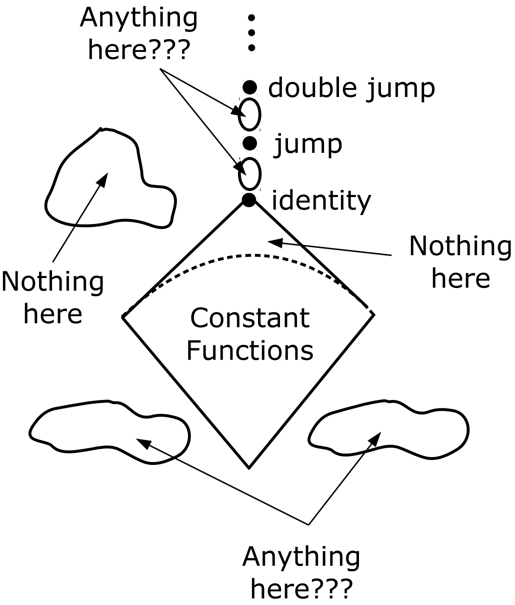
Picture of functions on the Turing degrees



Picture of functions on the Turing degrees



Picture of functions on the Turing degrees



Some measure theory background

If μ is a measure on X and $f : X \to Y$ is a function

Definition: The pushforward of μ by f is the measure given by $f_*\mu(A) = \mu(f^{-1}(A))$

Definition: f is measure preserving if $X = Y$ and $f_*\mu = \mu$

If μ and ν are ultrafilters on X

Definition (Rudin-Keisler order): $\mu \leq_{RK} \nu$ if there is some $f : X \to X$ such that $f_*\nu = \mu$

Principal ultrafilters are $\leq_{RK}$-minimal. If X is an ordinal then the normal ultrafilters are $\leq_{RK}$-minimal above the principal ultrafilters

Martin's conjecture and the Rudin-Keisler order

Fact: A function f on the Turing degrees is measure preserving if and only if

$$f_*(\text{Martin measure}) = \text{Martin measure}$$

Corollary: Part 1 of Martin's conjecture holds iff for all nonprincipal ultrafilters μ on the Turing degrees

- μ is not strictly below Martin measure in the Rudin-Keisler order
- and if μ is Rudin-Keisler equivalent to Martin measure then μ is Martin measure

Question: Show that no nonprincipal ultrafilter on the Turing degrees is strictly below Martin measure

Order preserving $\implies$ measure preserving

Theorem (L. and Siskind): If f is an order preserving function on the Turing degrees then either f is constant on a cone or f is measure preserving

Idea of the theorem: Think of f as a function on 2^ω and use a dichotomy theorem implied by AD

Dichotomy: AD implies that either range(f) is countable or contains a perfect set

Countable $\implies$ constant on a cone

Contains perfect set $\implies$ measure preserving

Order preserving $\implies$ measure preserving

Contains perfect set $\implies$ measure preserving

Main idea: Use a basis theorem for perfect sets

Theorem (L.): If $A \subseteq 2^\omega$ is a perfect set, B is a countable dense subset of A and x computes every element of B then for any y there are z_0, z_1, z_2, z_3 in A such that $x \oplus z_0 \oplus z_1 \oplus z_2 \oplus z_3 \geq_T y$

Strengthens a theorem of Groszek and Slaman

Useful property of range(f)**:** Every countable subset has an upper bound

An interesting consequence of our result

Theorem (Siskind): Suppose f and g are functions on the Turing degrees which are above the identity on a cone and which satisfy the following condition

$$\forall x, y\, (f(x) \equiv_T f(y) \implies g(x) \equiv_T g(y)) \qquad (*)$$

then f is below g on a cone

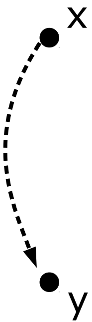
What's the point? Recall that part 2 of Martin's conjecture claims that $\leq_M$ is a well-order on functions above the identity. This result indicates it could at least be a total order

An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$.
By $(*)$ it doesn't matter which y is chosen

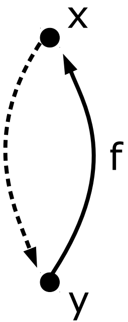
An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$. By $(*)$ it doesn't matter which y is chosen



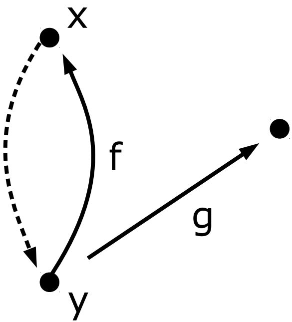
An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$.
By $(*)$ it doesn't matter which y is chosen



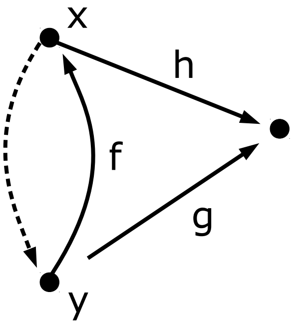
An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$.
By $(*)$ it doesn't matter which y is chosen



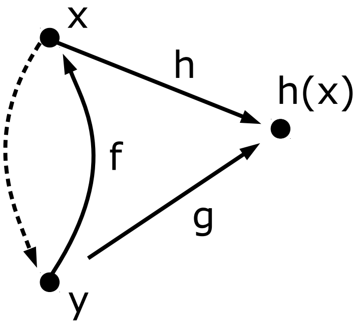
An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$. By $(*)$ it doesn't matter which y is chosen



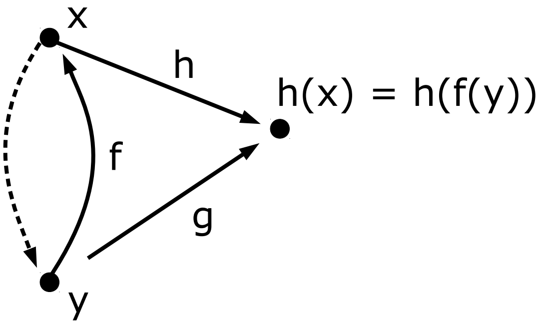
An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$. By $(*)$ it doesn't matter which y is chosen



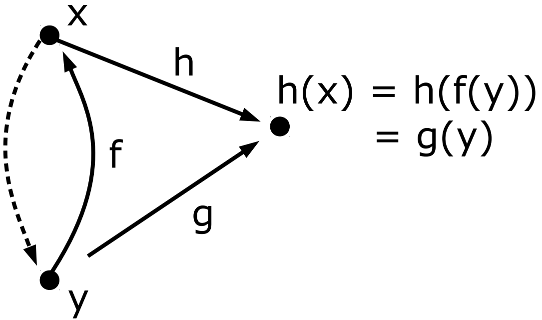
An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$.
By $(*)$ it doesn't matter which y is chosen



An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$.
By $(*)$ it doesn't matter which y is chosen



An interesting consequence of our result: proof

Proof (sketch): Define h as follows: start with x and find some y such that $f(y) = x$. Set $h(x) = g(y)$. By $(*)$ it doesn't matter which y is chosen

Key point: can show that h is measure-preserving and $h \circ f = g$

Since h is above the identity on a cone, g is above f on a cone. Intuitively $h = g - f$ and h is "positive"

Martin's conjecture for measure preserving functions

Theorem (L. and Siskind): Part 1 of Martin's conjecture holds for measure preserving functions

Theorem, restated: Every measure preserving function is above the identity on a cone

How does the proof work?

Overview of the proof

(1) General framework is a basic topological fact about continuous, injective functions

(2) To apply this in our case, study the structure of measure-preserving functions under determinacy

(3) In more detail: use determinacy to get certain auxiliary functions associated to a measure preserving function (basically a Skolem function witnessing that it is measure preserving, and the inverse of this Skolem function)

A basic fact of topology

A basic theorem in topology: If $F : X \to X$ is a continuous, injective function on a compact, Hausdorff topological space X then F has continuous inverse range$(F) \to X$.

Computability theory version: If $F : 2^\omega \to 2^\omega$ is a computable injective function then for all x, $F(x)$ can compute x.

Key idea: To show a function f is above the identity, it is enough to find a computable, injective function which is below f.

Examining the definition of measure preserving

Definition: A function f on the Turing degrees is measure preserving if

$$\forall x\, \exists y\, (\forall z\, (z \geq_T y \implies f(z) \geq_T x))$$

This definition naturally suggests looking at a Skolem function which witnesses that f is measure preserving

"f is going to infinity in the limit, but how fast?"

Modulus of a measure preserving function

Definition: If f is a measure preserving function, a modulus for f is a function g such that for all x,

$$z \geq_T g(x) \implies f(z) \geq_T x$$

Call g an increasing modulus for f if $g(x) \geq_T x$

It may seem obvious that every measure-preserving functions has a modulus. But you are probably using the axiom of choice

However, it is also true under determinacy! (By a uniformization theorem)

Disclaimer: needs $\mathsf{AD}_{\mathbb{R}}$ or AD^+

How to use the modulus

Remember: We are trying to find a computable injective function which f computes. Here's how we find it.

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g (like in the jump inversion via nuclear flyswatter example)

How to use the modulus

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g

Explanation: Suppose $h : \text{range}(g) \to 2^\omega$ is an inverse for g—i.e. $g(h(x)) = x$.

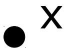
- h is injective: if $h(x) = h(y)$ then

$$x = g(h(x)) = g(h(y)) = y$$

- h is computable: $h(x) \leq_T g(h(x)) = x$ because g is increasing
- f is above h: $x \geq_T g(h(x))$ so $f(x) \geq_T h(x)$

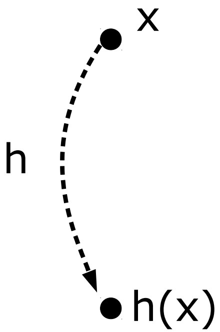
How to use the modulus

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g



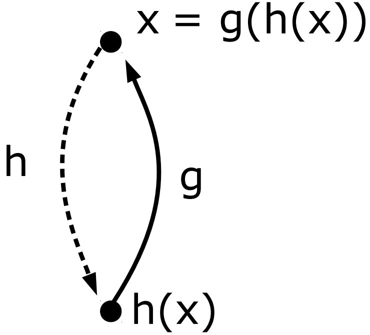
How to use the modulus

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g



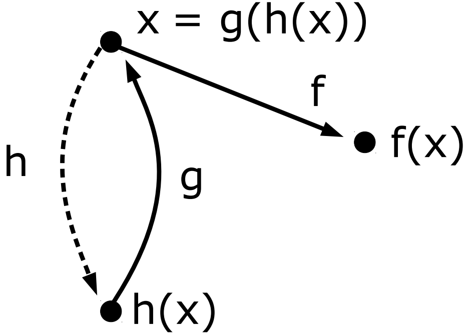
How to use the modulus

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g



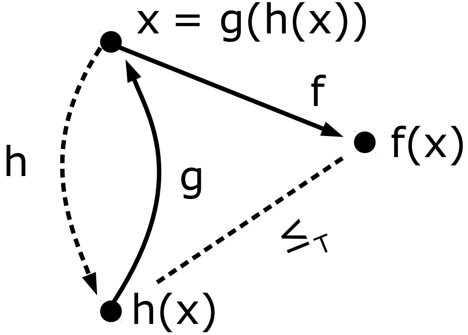
How to use the modulus

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g



How to use the modulus

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g



How is this possible??

This function h seems like exactly the kind of thing that's supposed to be ruled out by Martin's conjecture! It's decreasing and injective (so it's not constant on any cone). Why is this possible?

Answer: h is actually a function on 2^ω and there is no guarantee it is Turing invariant (i.e. even if $x \equiv_T y$ we may have $h(x) \not\equiv_T h(y)$) so it does not induce a function on Turing degrees.

This is a key point in our proof: you can study a Turing invariant function by relating it to a non-Turing invariant function you get with determinacy

Speculations

Question 1: Are there more uses of measure preserving functions?

The concept of a measure preserving function on the Turing degrees seems very natural and helped us discover our current proof of part 1 of Martin's conjecture for order preserving functions

Benny's proof is another example of a use of measure preserving functions in studying Martin's conjecture. Are there more uses?

Speculations

Question 2: Can thinking about the Rudin-Keisler order on ultrafilters on the Turing degrees help solve part 1 of Martin's conjecture?

The Rudin-Keisler order on ordinals and facts about which nonprincipal ultrafilters are minimal in it play a role in set theory. Is there a similar theory here?

The statement about the Rudin-Keisler order offers a novel way to decompose part 1 of Martin's conjecture into two smaller parts. Could proving one or both of these parts individually be more tractable than proving part 1 of Martin's conjecture?

Speculations

Question 3: Are there more uses for combining basis theorems for perfect sets with the basic topological fact I mentioned earlier?

I also used these two ingredients to prove the regressive case of the hyperarithmetic version of Martin's conjecture and to prove a statement about the arithmetic version of Martin's conjecture. Are there more uses?

Speculations

Question 4: Does changing perspective to the ultrapower by the Martin measure give any additional insight?

This one requires a little more explanation...

The view from the ultrapower

Under AD, Martin measure is an ultrafilter, so we can take ultrapowers by it

For instance, we can take the ultrapower of the partial order of Turing degrees itself

A representative of an element of this ultrapower is just a function from the Turing degrees to the Turing degrees. The ordering on these functions induced by the ultrapower is just the Martin ordering. And the equivalence relation induced by the ultrapower is Martin equivalence. The picture of the Martin order from earlier is also a picture of this ultrapower.

The view from the ultrapower

A measure preserving function f on the Turing degrees has a dual life

On the one hand, it is a representative of the ultrapower of the Turing degrees by the Martin measure

On the other hand, it induces a map from this ultrapower to itself as follows

$$[g] \mapsto [g \circ f]$$

We found it helpful to study a similar map on the ultrapower of the ordinals by Martin measure

The view from the ultrapower

There are some hints that studying ultrapowers by the Martin measure could be helpful

This ultrapower, along with a related structure called the "generic ultrapower," have been used in other parts of set theory where the axiom of determinacy plays a role. Could techniques from those areas be helpful in studying Martin's conjecture?

Is this related to our observations about Martin's conjecture and the Rudin-Keisler order on ultrafilters on the Turing degrees?