Encoding information into dense sets

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Joint work with Matthew Harrison-Trainor

Problem. Given a set X, find an infinite set $A \subseteq \mathbb{N}$ such that all infinite subsets of A compute X.

Classic solution. Think of A as a subset of $2^{<\omega}$ and let A be the set of initial segments of X

$$X = 10110100\ldots$$

A	⊙	•	⊙	•	•	⊙	•	•	•	•	•	•	⊙	•	•	$\cdots$
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
		0	1	00	01	10	11	000	001	010	011	100	101	110	111	

As a subset of $\mathbb{N}$, A is sparse.

Number of elements of A less than $n \approx \log(n)$

Question. Given a set X is there always a dense set $A \subseteq \mathbb{N}$ such that all infinite subsets of A compute X?

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To answer this question, we need to decide what "dense" means.

Definition. Given a set $A \subseteq \mathbb{N}$,

$$\text{lower density of } A = \underline{\rho}(A) = \liminf_{n \to \infty} \frac{|A \cap [n]|}{n+1}$$

$$\text{upper density of } A = \overline{\rho}(A) = \limsup_{n \to \infty} \frac{|A \cap [n]|}{n+1}$$

For any uncomputable X,

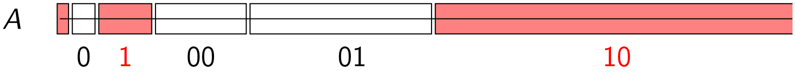
Theorem (Harrison-Trainor and L.). For any set $A \subseteq \mathbb{N}$ of positive lower density, A has an infinite subset which does not compute X.

Theorem (anyone in this room). There is a set A of positive upper density such that every infinite subset of A computes X.

Theorem (anyone in this room). For any X, there is a set A of positive upper density such that every infinite subset of A computes X.

Proof.

$$X = 10110100\ldots$$



Theorem (Harrison-Trainor and L.). For any uncomputable X and set $A \subseteq \mathbb{N}$ of positive lower density, A has an infinite subset which does not compute X.

Key ingredients:

(1) Mathias forcing

(2) Seetapun's theorem[1]. For every uncomputable set X and set $A \subseteq \mathbb{N}$, there is an infinite subset of either A or $\overline{A}$ which does not compute X.

[1] First explicitly proved by Dhzafarov and Jockusch and sometimes called "strong cone avoidance for RT^1_2."

Mathias forcing

End result of Mathias forcing: An infinite set $G \subseteq \mathbb{N}$

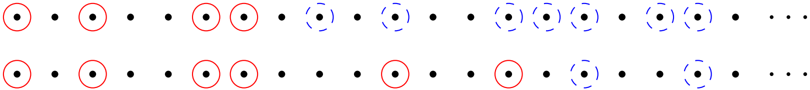
Conditions. Pairs (F, I) such that

- F is a finite subset of $\mathbb{N}$, called the stem of the condition
- I is an infinite subset of $\mathbb{N}$, called the reservoir of the condition

A condition (F, I) is a partial specification of G: elements of F have already been put into G and elements of I may be put into G later

Extension of conditions. (F, I) is extended by (F', I') if

- $F \subseteq F' \subseteq F \cup I$
- and $I \supseteq I'$

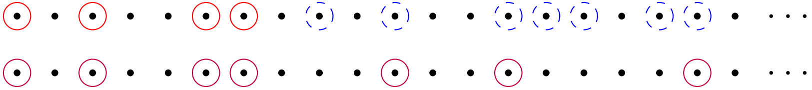


A condition (F, I) can be extended by choosing some elements of I to add to F and removing some elements from I

End result of Mathias forcing: An infinite set $G \subseteq \mathbb{N}$
Conditions. (F, I) specifying $F \subseteq G \subseteq F \cup I$
Extension. $(F, I) \geq (F', I')$ means $F \subseteq F' \subseteq F \cup I$ and $I' \subseteq I$

For any filter $\mathcal{G}$ for Mathias forcing, define $G = \bigcup_{(F,I) \in \mathcal{G}} F$
G is the set being specified by the Mathias conditions in $\mathcal{G}$

Definition. A set $B \subseteq \mathbb{N}$ is compatible with a condition (F, I) if $F \subseteq B \subseteq F \cup I$.



The point. A condition (F, I) says that the set G being built is compatible with (F, I). So to ensure G has some property, we can try to ensure *all* sets compatible with (F, I) have that property.

Goal. An infinite subset of A which does not compute X

General strategy. Pick a generic filter $\mathcal{G}$ and let $G = \bigcup_{(F,I)\in\mathcal{G}} F$. For each n, define

$$D_n = \{(F, I) \mid |F| \geq n\}$$
$$E_n = \{(F, I) \mid \text{for all } B \text{ compatible with } (F, I),\ \Phi_n(B) \neq X\}.$$

To show that:

(1) ***G* ⊆ *A*:** make sure $(\varnothing, A) \in \mathcal{G}$.

(2) ***G* is infinite:** show that for each n, D_n is dense.

(3) ***G* does not compute *X*:** show that for each n, E_n is dense.

Alternative view. Choose a sequence

$$(\varnothing, A) = (F_0, I_0) \geq (F_1, I_1) \geq (F_2, I_2) \geq \dots$$

so that each (F_{n+1}, I_{n+1}) is in both D_n and E_n. Define $G = \bigcup_n F_n$.

Approach 1: Pure Mathias forcing

Theorem (Harrison-Trainor and L.). For any uncomputable X and set $A \subseteq \mathbb{N}$ of positive lower density, A has an infinite subset which does not compute X.

Strategy. Pick a Mathias generic G compatible with $(\varnothing, A)$ and show that for each n,

$$D_n = \{(F, I) \mid |F| \geq n\}$$
$$E_n = \{(F, I) \mid \text{for all } B \text{ compatible with } (F, I),\ \Phi_n(B) \neq X\}$$

are both dense.

It's easy to pick G compatible with $(\varnothing, A)$

It's easy to show each D_n is dense.

Problem. E_n is not always dense.

Suppose every infinite subset of I computes X via Φ_n.
No extension of $(\varnothing, I)$ is in E_n: If $(\varnothing, I) \geq (F', I')$ then$B = F' \cup I'$ is compatible with $(F', I')$and $\Phi_n(B) = X$.

Solution. Restrict which sets are allowed to be reservoirs.

Approach 2: Mathias forcing with dense reservoirs

Goal: Given A of positive lower density and X uncomputable, find an infinite $G \subseteq A$ which doesn't compute X

Approach. Use Mathias forcing and show that for each n, the set

$$E_n = \{(F, I) \mid \text{for all } B \text{ compatible with } (F, I),\ \Phi_n(B) \neq X\}$$

is dense.

Problem. E_n is not always dense.

Solution. Restrict which sets are allowed to be reservoirs. Try to forbid sets whose infinite subsets all compute X

Natural idea. Require reservoirs to have positive lower density.

Problem. E_n is still not dense.

Claim. In Mathias forcing where reservoirs are required to have positive lower density, there is some n such that the set

$$E_n = \{(F, I) \mid \text{for all } B \text{ compatible with } (F, I),\ \Phi_n(B) \neq X\}$$

is not dense.

Fact. There is a set I of positive lower density such that all subsets of I of positive lower density compute X uniformly.

Proof of claim.Let I be as in the fact and suppose all subsets of I of positive lower density compute X via Φ_n.

No extension of $(\varnothing, I)$ is in E_n.

If (F', I') extends $(\varnothing, I)$ and I' has positive lower density then $B = F' \cup I'$ is compatible with (F', I') and has positive lower density, hence $\Phi_n(B) = X$

Approach 3: Mathias forcing with somewhat dense reservoirs

Goal: Given A of positive lower density and X uncomputable, find an infinite $G \subseteq A$ which doesn't compute X

Approach. Use Mathias forcing and show that for each n,

$$E_n = \{(F, I) \mid \text{for all } B \text{ compatible with } (F, I),\ \Phi_n(B) \neq X\}$$

is dense.

Problem. E_n is not always dense:

(1) When there are no restrictions on the reservoirs

(2) When reservoirs are required to have positive lower density

In (1), there are too many possible reservoirs: we could have (F, I) where all infinite subsets of I compute X.

In (2), there are too few possible reservoirs: for a given (F, I), we may not be able to "thin out" I enough to witness that not all of its infinite subsets compute X

We want something in the middle.

Definition. A set $A \subseteq \mathbb{N}$ is

- δ-dense at n if
$$\frac{|A \cap [n]|}{n+1} \geq \delta.$$
- δ-dense along $D \subseteq \mathbb{N}$ if for all $n \in D$, A is δ-dense at n.
- dense along D if A is δ-dense along D for some $\delta > 0$.

Observations.

- A has positive lower density if and only if A is dense along $\mathbb{N}$
- A has positive upper density if and only if A is dense along some infinite D

The point. If A is δ-dense along D, think of δ and D as witnessing the positive upper density of A.

Note that "A has positive upper density" is Σ^0_3, but "A is δ-dense along D" is Π^0_1

Goal: Given A of positive lower density and X uncomputable, find an infinite $G \subseteq A$ which doesn't compute X

Approach. Use Mathias forcing and show that for each n,

$$E_n = \{(F, I) \mid \text{for all } B \text{ compatible with } (F, I),\ \Phi_n(B) \neq X\}$$

is dense.

Definition. A Mathias condition (F, I) is a density Mathias condition if there is some infinite set D such that I is dense along D and D does not compute X

Restricting to density Mathias conditions works!

$(\varnothing, A)$ is a density Mathias condition as witnessed by $\mathbb{N}$.

Key Lemma. For any n, the set E_n above is dense for density Mathias forcing.

Enter Seetapun's theorem

Goal: Given A of positive lower density and X uncomputable, find an infinite $G \subseteq A$ which doesn't compute X

Definition. A Mathias condition (F, I) is a density Mathias condition if there is some infinite set D such that I is dense along D and D does not compute X

Strategy. Use density Mathias forcing and show that for each n,

$$E_n = \{(F, I) \mid \text{for all } B \text{ compatible with } (F, I),\ \Phi_n(B) \neq X\}$$

is dense.

Useful lemma. If A has positive lower density and $A = A_0 \cup A_1$ then at least one of $(\varnothing, A_0)$ and $(\varnothing, A_1)$ is a density Mathias condition.

Note: It is easy to find sets A_0, A_1 such that $A_0 \cup A_1$ has positive lower density but neither A_0 nor A_1 do.

This explains why we use sets of positive upper density as reservoirs when the statement only mentions positive lower density.

Definition. A Mathias condition (F, I) is a density Mathias condition if there is some infinite set D such that I is dense along D and D does not compute X

Useful lemma. If A has positive lower density and $A = A_0 \cup A_1$ then at least one of $(\varnothing, A_0)$ and $(\varnothing, A_1)$ is a density Mathias condition.

Seetapun's theorem. For any uncomputable X and set $A \subseteq \mathbb{N}$, there is an infinite subset of either A or $\overline{A}$ which does not compute X.

Proof of lemma. Pick $\delta > 0$ such that A is δ-dense at every n.

Define $B = \{n \mid A_0 \text{ is } \delta/2\text{-dense at } n\}$.

Note that if $n \notin B$ then A_1 must be $\delta/2$-dense at n.

Seetapun's theorem $\implies$ either B or $\overline{B}$ contains an infinite subset D which does not compute X.

$D \subseteq B \implies (\varnothing, A_0)$ is a density Mathias condition
$D \subseteq \overline{B} \implies (\varnothing, A_1)$ is a density Mathias condition.