(a) Find all eigenvectors of \( \frac{d^3}{dx^3} - 9 \frac{d^2}{dx^2} + 27 \frac{d}{dx} \) with eigenvalue 27. [Hint: the polynomial \( t^3 - 9t^2 + 27t - 27 \) can be factored as \( (t - 3)^3 \).]

\[ f(x) \text{ is an eigenvector of } \frac{d^3}{dx^3} - 9 \frac{d^2}{dx^2} + 27 \frac{d}{dx} \text{ with eigenvalue 27 if and only if } \]

\[ \frac{d^3 f(x)}{dx^3} - 9 \frac{d^2 f(x)}{dx^2} + 27 \frac{d f(x)}{dx} = 27 f(x). \]

In other words, if and only if \( f(x) \) is a solution to the ODE:

\[ y''' - 9y'' + 27y' - 27y = 0 \]

Auxiliary equation: \( r^3 - 9r^2 + 27r - 27 = 0 \)

\( (r - 3)^3 = 0 \) (by the hint)

Solutions: \( c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} \)

where \( c_1, c_2, c_3 \) are any scalars, not all 0 (since eigenvectors are supposed to be nonzero).

(b) Use your answer to part (a) to find a nontrivial solution to the following PDE.

\[ \frac{\partial f}{\partial t} = \frac{\partial^3 f}{\partial x^3} - 9 \frac{\partial^2 f}{\partial x^2} + 27 \frac{\partial f}{\partial x} \]

Let \( T \) be the linear transformation \( \frac{\partial^3}{\partial x^3} - 9 \frac{\partial^2}{\partial x^2} + 27 \frac{\partial}{\partial x} \).

So the PDE is

\[ \frac{\partial f}{\partial t} = T(f). \]

We saw in class that if \( y(x) \) is an eigenvector of \( T \) with eigenvalue \( \lambda \) then \( e^{\lambda t} y(x) \) is a solution to the above PDE.

So by part (a), \( e^{3t} (c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}) \)

is a solution for any \( c_1, c_2, c_3 \in \mathbb{R} \) (and it is a nontrivial solution when \( c_1, c_2, c_3 \) are not all 0).