The Heat Equation

1. Let \( f(x,t) = e^{xt} \sin(x) + t^2 + 5 + \cos(x) \). Find \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \) and \( \frac{\partial^2 f}{\partial t^2} \).

2. Write down the definition of eigenvector of a linear transformation and then write down what it means for a function \( z(x) \) to be an eigenvector with eigenvalue \( \lambda \) of the linear transformation \( \frac{d^2}{dx^2} \).

3. Find all eigenvectors of \( \frac{d^2}{dx^2} \) with eigenvalue \(-9\).

4. Find all eigenvectors of \( \frac{d^2}{dx^2} \) with eigenvalue \(0\).

5. Find a solution to the following differential equation.

\[
\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}; \quad u(0,t) = u(0,L) = 0; \quad u(x,0) = \sin(\pi x/L) + 13 \sin(5\pi x/L)
\]

6. Find solutions to the heat equation with the boundary values given below. Try to use the same method we used earlier in class.

\[
\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0.
\]

Definitions and Theorems

Definitions:

- Partial derivative
- PDE (partial differential equation)
- Heat Equation

Theorems:

- How to find solutions to the heat equation (which corresponds to finding eigenvectors of the second derivative).

Most important idea today: Finding solutions to linear PDEs like the heat equation uses exactly the same ideas we used to solve systems of linear ODEs: it’s all about finding eigenvectors of some linear transformation.