Review

1. Find a solution to the following initial value problem.
\[ y'' + 2y' + 5y = 26e^{2t}; \quad y(0) = 3; \quad y'(0) = 9 \]

Wronskian

1. (a) Find the Wronskian of 1 and \( e^{t^2} \).
   (b) Are 1 and \( e^{t^2} \) linearly independent? (Hint: Use part (a))
   (c) Is there any linear ODE for which both 1 and \( e^{t^2} \) are solutions?

2. (a) Find the Wronskian of \( t|t| \) and \( t^2 \).
   (b) Are \( t|t| \) and \( t^2 \) linearly independent?

Systems of ODEs

1. Reduce the following higher order ODE to a system of first order ODEs and then put that system in normal form.
\[ y''' + e^t y'' - \cos(t)y = 17 \]

2. Reduce the following system of higher order ODEs to a system of first order ODEs and then put that system in normal form.
\[ y'' = 5y' - 6z' + z + \sin(t) \]
\[ z'' = y' + z + 2 \]

3. Find the derivative of the following vector-valued functions.
   (a) \( \mathbf{f}(t) = \begin{bmatrix} \sin(t) \\ t^2 + te^{5t} \end{bmatrix} \)
   (b) \( y(t) = e^{5t} \mathbf{v} \) where \( \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \)

4. Check if each function given below is a solution to \( y' = Ay \).
   \[ A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \]
   (a) \( \mathbf{f}(t) = e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \)
   (b) \( \mathbf{g}(t) = \begin{bmatrix} \sin(t) \\ \frac{2}{3}e^{5t} \end{bmatrix} \)

5. Suppose \( X(t) \) is a fundamental matrix for the system \( y' = Ay \). Solve the initial value problem \( y' = Ay; \quad y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)
\[ X(t) = \begin{bmatrix} e^{3t} & 2e^{7t} \\ 5e^{3t} & 7e^{7t} \end{bmatrix} \]
Definitions and Theorems

Definitions:

- Pre-Wronskian
- Wronskian
- System of ODEs
- Vector of functions (equivalently a function from $\mathbb{R}$ to $\mathbb{R}^n$)
- Normal Form
- Fundamental matrix

Theorems:

- The Wronskian Lemma: Suppose $y_1, \ldots, y_n$ are solutions to a linear ODE. Then the Wronskian $W[y_1, \ldots, y_n]$ is non-zero everywhere if $y_1, \ldots, y_n$ are linearly independent and otherwise it is zero everywhere.

  Caution: This is not true if $y_1, \ldots, y_n$ are not all solutions to the same linear ODE. For arbitrary functions, if the Wronskian is nonzero at any point then they are linearly independent, but there are linearly independent functions whose Wronskian is zero everywhere.

- The initial value problem for a system of first order linear ODEs always has a solution and that solution is always unique.

- (If we have time) If $u$ is an eigenvector of $A$ with eigenvalue $r$ then $y(t) = e^{rt}u$ is a solution to $y'(t) = Ay(t)$.

Most important idea today: Every higher order linear ODE can be reduced to a system of first order ODEs.

Most important idea today if we have time to get to it: To find solutions to $y'(t) = Ay(t)$, find eigenvectors of $A$. Moral: any time you see a linear transformation, its eigenvectors are probably important!!