Review

1. Write a homogeneous differential equation such that $t^2e^{3t} + 5$ is a solution.
2. Write a homogeneous differential equation such that $e^{-2t} \cos(4t) + e^t$ is a solution.

Nonhomogeneous ODEs

1. Find a solution to the following ODEs.
   
   \begin{align*}
   & (a) \ y'' - y' + y = 4t^2 + 8 \\
   & (b) \ 2y'' - y' = 5 \cos(t)
   \end{align*}

2. You may assume without checking that $t^3 e^{-t}$ is a solution to $y''' + 3y'' + 3y' + y = 6e^{-t}$, that \( \sin(t) \) is a solution to $y''' + 3y'' + 3y' + y = -2 \sin(t) + 2 \cos(t)$. Find a solution to the following ODEs.
   
   \begin{align*}
   & (a) \ y''' + 3y'' + 3y' + y = e^{-t} \\
   & (b) \ y''' + 3y'' + 3y' + y = e^{-t} + \sin(t) - \cos(t)
   \end{align*}

3. Find the general solution to the following ODEs.
   
   \begin{align*}
   & (a) \ 2y'' - y' = 5 \cos(t). \\
   & (b) \ y'' + 2y' + 5y = 26e^{2t}.
   \end{align*}

4. Find a solution to the following initial value problems.
   
   \begin{align*}
   & (a) \ 2y'' - y' = 5 \cos(t); \quad y(0) = 3; \quad y'(0) = 0. \\
   & (b) \ y'' + 2y' + 5y = 26e^{2t}; \quad y(0) = 3; \quad y'(0) = 9.
   \end{align*}

Wronskian

1. (a) Find the Wronskian of 1 and $e^{t^2}$.
   (b) Are 1 and $e^{t^2}$ linearly independent? (Hint: Use part (a))
   (c) Is there any linear ODE for which both 1 and $e^{t^2}$ are solutions?

2. (a) Find the Wronskian of $t|t|$ and $t^2$.
   (b) Are $t|t|$ and $t^2$ linearly independent?
Definitions and Theorems

Definitions:

- Pre-Wronskian
- Wronskian

Theorems:

- If $y_1$ is a solution to the ODE $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = f_1$ and $y_2$ is a solution to the ODE $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = f_2$ then $c_1 y_1 + c_2 y_2$ is a solution to $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = c_1 f_1 + c_2 f_2$. The textbook calls this the “superposition principle” but it is really just part of the definition of ‘linear transformation.’

- If $y_p$ is a solution to the ODE $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = f$ and the general solution of the homogeneous ODE $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0$ is $c_1 y_1 + \ldots + c_n y_n$ then $y_p + c_1 y_1 + \ldots + c_n y_n$ is the general solution to $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = f$. This is really just a statement about linear transformations that we first saw in chapter 1, section 5 of the linear algebra textbook.

- (If we have time) The Wronskian Lemma: Suppose $y_1, \ldots, y_n$ are solutions to a linear ODE. Then the Wronskian $W[y_1, \ldots, y_n]$ is nonzero everywhere if $y_1, \ldots, y_n$ are linearly independent and otherwise it is zero everywhere.

  Caution: This is not true if $y_1, \ldots, y_n$ are not all solutions to the same linear ODE. For arbitrary functions, if the Wronskian is nonzero at any point then they are linearly independent, but there are linearly independent functions whose Wronskian is zero everywhere.

Most important idea today: If $T$ is a linear transformation then the set of solutions to $T(x) = b$ is just the kernel of $T$ translated by some vector and therefore to find all solutions to a nonhomogeneous linear ODE it is enough to find one solution to the nonhomogeneous ODE and all solutions to the corresponding homogeneous ODE.