Spectral Theorem

1. Is the matrix $A$ orthogonally diagonalizable? If so, find a diagonal matrix $D$ and an orthogonal matrix $P$ such that $A = PDP^T$.

\[ A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \]

2. Suppose $A$ is a symmetric matrix. Show that $A$ is positive semi-definite if and only if all eigenvalues of $A$ are nonnegative.

3. Show that if $A$ is any matrix, $A^TA$ is symmetric and positive semi-definite.

4. (This one’s kind of tricky.) Show that if $A$ is a symmetric matrix then the largest eigenvalue of $A$ is the maximum possible value of

\[ \frac{x \cdot (Ax)}{x \cdot x} \]

(Hint: what happens to the expression above if you pick an orthonormal basis of eigenvectors for $A$ and try writing some $x$ in terms of that basis?)

Definitions and Theorems

Definitions and examples:

- Symmetric matrix. Also known as self-adjoint (especially in math) or Hermitian (especially in physics) although these mean something slightly different than symmetric when there are complex numbers involved.

- Quadratic Form.

- Positive definite matrix, positive semi-definite matrix (PSD).

Theorems:

- The Spectral Theorem: If $A$ is an $n \times n$ symmetric matrix then $A$ has only real eigenvalues and there is an orthogonal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$ (i.e. $A$ is orthogonally diagonalizable).

Most important idea today: THE SPECTRAL THEOREM! (Having a basis of orthogonal eigenvectors is super useful.)