**Review**

1. What is \( \{0\}^\perp \)? (Assume \( 0 \) here is referring to the zero vector in \( \mathbb{R}^n \).)

2. What is \((\mathbb{R}^n)^\perp\)?

3. If \( W \) is a subspace of \( \mathbb{R}^n \), what is \( W \cap W^\perp \)?

4. Show that for any vector \( v \) in \( \mathbb{R}^n \), \( v \cdot v \geq 0 \). When is it exactly equal to zero?

5. **Super useful fact:** Show that if \( u \) is orthogonal to the vectors \( v_1, \ldots, v_m \) then it is orthogonal to every vector in \( \text{span}\{v_1, \ldots, v_m\} \).

**Transpose and Orthogonal Complement**

1. Show that for any \( n \times m \) matrix \( A \), \( \text{Col}(A)^\perp = \text{Null}(A^T) \).

2. Show that for any \( n \times m \) matrix \( A \), \( \dim(\text{Row } A) = \dim(\text{Col } A) \).

3. Show that if \( W \) is a subspace of \( \mathbb{R}^n \), \( \dim(W^\perp) = n - \dim(W) \). (Hint: think of \( W \) as the column space of some matrix.)

4. Show that \((W^\perp)^\perp = W \).

**Orthogonal Basis**

1. Suppose \( W \) is a subspace of \( \mathbb{R}^n \) and \( \{v_1, \ldots, v_m\} \) is a basis for \( W \). Also suppose \( u \) is in \( W \) and that \( u = c_1 v_1 + \ldots + c_m v_m \). If \( 1 \leq i \leq m \), what is \( u \cdot v_i \)?

2. Solve the following system of linear equations without doing any row reduction. (Hint: the columns of the coefficient matrix are orthogonal to each other.)
   
   \[
   \begin{align*}
   x_1 &+ 6x_2 &+ 2x_3 &= 23 \\
   2x_1 &- x_2 &+ x_3 &= 1 \\
   3x_1 & &- 16x_3 &= -29 \\
   4x_1 &- x_2 &+ 11x_3 &= 23
   \end{align*}
   \]

**Definitions and Theorems**

**Definitions:**

- Orthogonal
- Orthogonal complement
- Transpose
- Row Space
- Orthogonal Set, Orthogonal Basis
- Orthonormal Set, Orthonormal Basis
- Projection onto a subspace (i.e. \( \text{proj}_W(u) \))
Theorems:

- If a vector is orthogonal to every vector in a list then it is also orthogonal to all vectors in the span of that list.
- Col(A)⊥ = Null(Aᵀ)
- rank(A) = rank(Aᵀ)
- An orthogonal set of nonzero vectors is linearly independent.

For any subspace W of \( \mathbb{R}^n \), every vector can be written in a unique way as a sum of a vector in W and a vector in W⊥. The first vector in this sum is the closest vector in W to the original vector.

Most important idea today: Suppose you want to figure out how to write one vector as a linear combination of a list of vectors. When the vectors in the list are all orthogonal to each other, it is super easy to do.