The Matrix of a Linear Transformation

1. For each of the following linear transformations, find a basis for the domain and codomain and write the matrix of the linear transformation relative to those bases.

   (a) \( T : M_{2 \times 2} \rightarrow M_{2 \times 2} \) defined by \( T(B) = AB \) where
   \[
   A = \begin{bmatrix}
   1 & 5 \\
   2 & 6 
   \end{bmatrix}
   \]

   (b) \( T : \text{span}\{\sin(x), \cos(x)\} \rightarrow \mathbb{R}^3 \) defined by \( T(f) = \begin{bmatrix} f(0) \\ f(\pi) \\ f(\pi/2) \end{bmatrix} \).

   (c) \( T : \text{span}\{\sin(x), \cos(x)\} \rightarrow \mathbb{R} \) defined by \( T(f) = \int_{0}^{\pi} f(x) \, dx \).

2. Which of the linear transformations in the previous questions are one-to-one? Which ones are onto?

3. For each linear transformation from the previous question, find a basis for the kernel and for the range.

4. The following linear transformation is one-to-one and onto (you do not have to check this). Find its inverse.

   \( T : P_1 \rightarrow \mathbb{R}^2 \)
   \[
   T(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}
   \]

5. Explain why a degree one polynomial is uniquely determined by its values at 1 and 2. (Hint: look at the previous question.)

Change of Basis

For all the problems in this section, let \( \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \) and \( \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \).

1. (a) Is \( \mathcal{B} \) a basis for \( \mathbb{R}^2 \)? What about \( \mathcal{C} \)?

   (b) If \( \mathbf{v} \) is a vector in \( \mathbb{R}^2 \) such that \( [\mathbf{v}]_\mathcal{B} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) then what is \( \mathbf{v} \)?

   (c) If \( \mathbf{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \) then what is \( [\mathbf{u}]_\mathcal{B} \)?

2. Try to find a matrix \( A \) such that for any \( \mathbf{v} \in \mathbb{R}^2 \), \( A\mathbf{v} = [\mathbf{v}]_\mathcal{B} \).

3. If \( \mathbf{v} \) is a vector in \( \mathbb{R}^2 \) such that \( [\mathbf{v}]_\mathcal{B} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) then what is \( [\mathbf{v}]_\mathcal{C} \)?

4. Try to find a matrix \( B \) such that for any \( \mathbf{v} \in \mathbb{R}^2 \), \( B[\mathbf{v}]_\mathcal{B} = [\mathbf{v}]_\mathcal{C} \).
5. Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that
\[
T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]
Find the matrix of $T$ relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

6. With $T$ as in the previous question, find the standard matrix of $T$. 