Review

1. Are the following vectors in $\mathbb{R}^3$ linearly independent? Do they span all of $\mathbb{R}^3$?

\[
\begin{bmatrix}
0 \\
1 \\
3
\end{bmatrix}, \begin{bmatrix}
2 \\
-3 \\
4
\end{bmatrix}, \begin{bmatrix}
2 \\
-2 \\
7
\end{bmatrix}
\]

Linear Transformations

1. Which of the following are linear transformations?
   
   (a) $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(p) = \frac{dp}{dx}$.
   
   (b) $T : M_{2\times3} \to M_{4\times3}$ defined by $T(B) = AB$ where

   \[
   A = \begin{bmatrix}
   1 & 5 \\
   2 & 6 \\
   3 & 7 \\
   4 & 8
   \end{bmatrix}
   \]

   (c) $T : M_{2\times2} \to M_{2\times2}$ defined by $T(A) = A + I$.
   
   (d) $T : C([0,1]) \to C([0,1])$ defined by $T(f) = f^2$.
   
   (e) $T : C(\mathbb{R}) \to \mathbb{R}^2$ defined by $T(f) = \begin{bmatrix} f(0) \\ f(\pi) \end{bmatrix}$.
   
   (f) $T : C([0,1]) \to \mathbb{R}$ defined by $T(f) = \int_0^1 f(x) \, dx$.

2. For each linear transformation in the previous question, find a nonzero element of the range and a nonzero element of the kernel.

Why You Should Love Bases

1. Answer the following questions.

   (a) Is $\{1, x, x^2\}$ a basis for $\mathbb{P}_2$?
   
   (b) Is $\{1, x, x^2\}$ a basis for $\mathbb{P}_3$?
   
   (c) Is $\{1, x^2, x, 3x^2 - 2\}$ a basis for $\mathbb{P}_2$?
   
   (d) What is the dimension of $\mathbb{P}_2$? What about $\mathbb{P}_3$?

2. In each item below, you are given a vector space, a basis for the vector space and a vector in the vector space. Write the vector as a coordinate vector in terms of the basis (you do not have to check that the list of vectors is actually a basis).

   (a) $\mathbb{P}_3$, $\{1, x, x^2, x^3\}$, $3x^2 - 5$
   
   (b) $M_{2\times2}$, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, $\begin{bmatrix} 3 & -4 \\ 5 & 1 \end{bmatrix}$
(c) \( \text{span}\{\sin(x), \cos(x)\} \) (this is a subspace of \( C(\mathbb{R}) \)), \{\sin(x), \cos(x)\}, 4\cos(x) - \sin(x)

(d) \( \mathbb{P}_2 \), \{1 + x, x + x^2, x^2 + x^3, x^3 - 1\}, 1

(e) \( \mathbb{R}^2 \), \( \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \)

3. Find a polynomial \( p \) such that
\[
[p]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\]
where \( B = \{1, x, x^2 + x\} \).

4. Is \( \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} \) in \( \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 10 \end{bmatrix} \right\} \)?

5. Are the following polynomials linearly independent? Do they span all of \( \mathbb{P}_2 \)?
\[
3x^2 + x, 4x^2 - 3x + 2, 7x^2 - 2x + 2
\]

6. Do the following functions form a basis for \( \text{span}\{\sin(x), \cos(x), e^x\} \)? (You may assume without proof that \( \sin(x) \), \( \cos(x) \), and \( e^x \) are linearly independent.)
\[
\cos(x) + 3e^x, 2\sin(x) - 3\cos(x) + 4e^x, 2\sin(x) - 2\cos(x) + 7e^x
\]

7. What is the dimension of \( \text{span}\{1 + x, x + x^2, x^2 - 1\} \)?