① Find an example or explain why none exist:

A, B $n \times m$ matrices

same rank

there are no invertible matrices C and D s.t. $CAD = B$

No example exists

$$[A]_{\sim}^{\sim} = \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

Null(A)

basis for Null(A) $\left\{ v_1, v_2, \ldots, v_r \right\}$

$\left\{ v_{r+1}, \ldots, v_m \right\} \to \left\{ Av_{r+1}, \ldots, Av_m \right\}$ basis for Col(A)

$Av_{r+1}, \ldots, Av_m, x_1, \ldots, x_{n-m}$ basis for $\mathbb{R}^n$

Null(B)

basis for Null(B) $\left\{ u_1, u_2, \ldots, u_r \right\}$

$\left\{ u_{r+1}, \ldots, u_m \right\} \to \left\{ Bu_{r+1}, \ldots, Bu_m \right\}$ basis for col(B)

$Bu_{r+1}, \ldots, Bu_m, y_1, \ldots, y_{n-m}$

$u_1 \mapsto \vec{0}$

$\vdots$

$u_r \mapsto \vec{0}$

$u_{r+1} \mapsto Bu_{r+1}$

$\vdots$

$u_m \mapsto Bu_{m+1}$

② Example: A square matrix A and a number λ s.t. λ^2 is an eigenval of A^2 but λ is not an eigenval. of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

eigenvals: 1   eigenvals: 1

$\lambda = -1 \qquad \lambda^2 = 1$

λ^2 is an eigenval. of A^2

$$\Updownarrow$$

$$\det(A^2 - \lambda^2 I) = 0$$

$$\Updownarrow$$

$$\det((A - \lambda I)(A + \lambda I)) = 0$$

$$\Updownarrow$$

$$\det(A - \lambda I)\det(A + \lambda I) = 0$$

It is true that if λ is an eigenval. of A then λ^2 is an eigenval of A^2

It is also true that if λ^2 is an eigenval of A^2 then either λ or $-\lambda$ is an eigenval of A

True/False

① A, B $n\times n$ matrices **False**

$\vec{u}, \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n \Rightarrow \vec{u} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$

$\vec{u} = A\vec{v}_1 + B\vec{v}_2$

$\vec{u} = \begin{bmatrix}2\\2\end{bmatrix}$ $\quad A\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$

$A = B = \begin{bmatrix}1&1\\1&1\end{bmatrix}$ $\quad \vec{v}_1 = \vec{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}$

② A & B $n\times n$ matrices s.t. $AB = BA$ & $\vec{v}$ is an eigenvector of B & $A\vec{v} \neq \vec{0}$ $\Rightarrow$ $A\vec{v}$ is an eigenvector of B **True**

$B\vec{v} = \lambda\vec{v}$

$$B(A\vec{v}) = (BA)\vec{v} = (AB)\vec{v} = A(B\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v})$$

③ If $A^4 = I$ then the only possible eigenvalues of A are 1 & -1 **False**

$y^{(3)} - 3y'' + 4y' - 2y =$

$\chi_A(t) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$ $\quad A = \begin{bmatrix}0&-1\\1&0\end{bmatrix}$ $\quad A^4 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

④ A diagonalizable & 0 & 5 are its only eigenvals then $\text{Col}(A) = E_5$ **True**

$A\vec{v} = A(\vec{v}_1 + \dots + \vec{v}_i + \dots + \vec{v}_n) = 5\vec{v}_i + \dots$

⑤ There is no homogeneous linear ODE s.t. both e^t & $e^t\cos(t)$ are sol'ns **False**

$e^t\sin(t)$

$1+i \leftarrow e^t\cos(t)$ $\quad e^t \to 1$

$(\lambda - 1)(\lambda - (1+i))(\lambda - (1-i))$

$= (\lambda - 1)(\lambda^2 - 2\lambda + 2) = (\lambda^3 - 3\lambda^2 + 4\lambda - 2)$

⑥ Find 3 different 2×5 matrices whose singular values are 2 & 3

It's okay to express your answers as products of other matrices.

Singular values of A: square roots of eigenvals of A^TA

$A = U \Sigma V^T$

orthogonal matrices = columns orthonormal

SVD of A

singular values are "diagonal" entries of Σ

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{bmatrix}}_{2\times 5} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{5\times 5}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

7. $f: [-\pi, \pi] \to \mathbb{R}$

Fourier series of f is $3 + \sum_{n=1}^{\infty} \frac{1}{2n^2} \sin(nx)$

a) Find $\int_{-\pi}^{\pi} f(x)\,dx$

Fourier series of f is

$$\frac{\langle f, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle f, \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \sin(x) + \frac{\langle f, \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle} \cos(x) + \dots$$

$$\int_{-\pi}^{\pi} 1^2\,dx = 2\pi$$

$$\int_{-\pi}^{\pi} f(x) \cdot 1\,dx$$

$$3 = \frac{\int_{-\pi}^{\pi} f(x)\,dx}{2\pi} \Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = 6\pi$$

b) Find the Fourier series of $f(x) + \cos(x)$

$$3 + \cos(x) + \sum_{n=1}^{\infty} \frac{1}{2n^2} \sin(nx)$$

c) Find the Fourier series of $f(x)\cos(x)$

$$f(x) = 3\cos(x) + \frac{1}{16}\sin(x) + \sum_{n=2}^{\infty} \frac{1}{4}\left(\frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right) \sin(n$$

$$\left(3 + \sum_{n=1}^{\infty} \frac{1}{2n^2}\sin(nx)\right)\cos(x)$$

$$= \left(3\cos(x) + \sum_{n=1}^{\infty} \frac{1}{2n^2}\sin(nx)\cos(x)\right)$$

$$= \frac{1}{2}\left(\sin(nx+x) + \sin(nx-x)\right)$$

$$= \frac{1}{2}\left(\sin((n+1)x) + \sin((n-1)x\right)$$

$$= 3\cos(x) +$$

$$\left(\frac{1}{2}\left(\frac{1}{2}\left(\sin(2x) + \sin(0x)\right)\right)\right.$$

$$+ \frac{1}{8}\left(\frac{1}{2}\left(\sin(3x) + \sin(x)\right)\right)$$

$$\left. + \frac{1}{18}\left(\frac{1}{2}\left(\sin(4x) + \sin(2x)\right)\right) + \cdots \right)$$

coefficient of $\sin(nx) = \frac{1}{2(n-1)^2}\cdot\frac{1}{2} + \frac{1}{2(n+1)^2}\cdot\frac{1}{2}$

d) Find the function $g(x) \in \text{span}\{\sin(x), \sin(2x), \sin(3x)\}$

s.t. $\int_{-\pi}^{\pi} (f(x) - g(x))^2 \, dx$ is as small as possible

$\frac{1}{2}\sin(x) + \frac{1}{8}\sin(2x) + \frac{1}{18}\sin(3x)$

In $\mathbb{R}^n$, the vector $\vec{v}$ in $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ s.t.

$\|\vec{u} - \vec{v}\|^2$ is as small as possible is

$$\text{proj}_{\text{span}\{v_1, v_2, v_3\}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{u} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3$$

if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ orthogonal

$$\langle f - g, f - g \rangle = \|f - g\|^2$$

$$\text{proj}_{\text{span}\{\sin(x), \sin(2x), \sin(3x)\}}(f)$$

Fourier coefficients

$$= \frac{\langle f, \sin(x) \rangle}{\langle \sin(x), \sin(x) \rangle}\sin(x) + \frac{\langle f, \sin(2x) \rangle}{\langle \sin(2x), \sin(2x) \rangle}\sin(2x) + \frac{\langle f, \sin(3x) \rangle}{\langle \sin(3x), \sin(3x) \rangle}\sin(3x)$$

④ $B = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -4/\sqrt{42} \\ 5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \right\}$

orthonormal basis for $\mathbb{R}^3$

If A is a square orthogonal matrix then $A^{-1} = A^T$

Find $\underset{B \leftarrow std}{P} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ -4/\sqrt{42} & 5/\sqrt{42} & 1/\sqrt{42} \\ 2/\sqrt{14} & 1/\sqrt{14} & 3/\sqrt{14} \end{bmatrix}$

$$\underset{B \leftarrow std}{P} \overbrace{[\vec{v}]_{std}}^{=\vec{v}} = [\vec{v}]_B \quad \text{for all } \vec{v} \in \mathbb{R}^3$$

$$\underset{B \leftarrow std}{P} \vec{v} = [\vec{v}]_B \qquad \underset{B \leftarrow std}{P}^{-1} = \underset{std \leftarrow B}{P}$$

columns are vectors in B

We want the inverse of this

$$\underset{std \leftarrow B}{P} = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} & 2/\sqrt{14} \\ 1/\sqrt{3} & 5/\sqrt{42} & 1/\sqrt{14} \\ -1/\sqrt{3} & 1/\sqrt{42} & 3/\sqrt{14} \end{bmatrix}$$

(5) $V = \text{span}\{\sin(x), \cos(x)\}$

$T: V \to V \quad T(f) = f'$

Find the eigenvalues of T

$i, -i$

$\sin(x), \cos(x)$ is a basis for V

$B = \{\sin(x), \cos(x)\}$

$${}_B[T]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$T(\sin(x)) = \cos(x) \rightsquigarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\cos(x)) = -\sin(x) \rightsquigarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Eigenvector for T:

$T(f) = \lambda f$

$\Updownarrow$

$f' = \lambda f$

$f' - \lambda f = 0$

Eigenvalues of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

char. polynomial: $\det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = (-\lambda)^2 - (-1)\cdot 1$

$= \lambda^2 + 1$

roots: $i, -i$

How to find SVD

$$A = \begin{bmatrix} \boxed{\vec{u}_1 / \|A\vec{v}_1\|} & \cdots \end{bmatrix} \begin{bmatrix} \boxed{\sqrt{\lambda_1} \quad \sqrt{\lambda_2}} & & 0 \\ & & \sqrt{\lambda_m} \\ 0 & & \\ 0 & 0\ 0\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boxed{— \vec{v}_1 —} \\ \vdots \\ — \vec{v}_m — \end{bmatrix}$$

A is an $n \times m$ matrix

① Find eigenvals of A^TA $\quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > \underset{0}{\overset{\shortparallel}{\lambda_{r+1}}} \cdots \lambda_m$

② Find an orthonormal basis of eigenvectors for A^TA

↳ for each eigenval. λ, find a basis for $E_\lambda = \text{Null}(A^TA - \lambda I)$ then use Gram-Schmidt to get an orthogonal basis & then normalize

$\vec{v}_1, \dots, \vec{v}_r, \dots, \vec{v}_m$

③ Use Gram-Schmidt to complete $A\vec{v}_1, \dots, A\vec{v}_r$ to an orthonormal basis for $\mathbb{R}^n$

$\overset{\shortparallel}{\vec{u}_1}, \dots, \overset{\shortparallel}{\vec{u}_r}, \vec{u}_{r+1}, \dots, \vec{u}_n$

& normalize to get unit vectors

SVD

$$A = \overset{n\times n}{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}} \overset{n\times m}{\begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ 0 & \ddots & \sqrt{\lambda_m} \\ 0 & 0 & 0 \end{bmatrix}} \overset{m\times m}{\begin{bmatrix} — & v_1 & — \\ & \vdots & \\ — & v_m & — \end{bmatrix}}$$

$A\vec{v}_{i \ldots m}$

square roots of eigenvals for A^TA

eigenvectors for A^TA

Reduced SVD

$$A = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_r} \end{bmatrix} \begin{bmatrix} — & v_1 & — \\ & \vdots & \\ — & v_r & — \end{bmatrix}$$

sq. roots of nonzero eigenvals of A^TA