① Which of these are subspaces?

a) The set of diff. functions $f: \mathbb{R} \to \mathbb{R}$ s.t. $f' = f$. Is a subspace

① $0' = 0$ ✓

② $(f+g)' = f' + g' = f + g$ ✓

③ $(af)' = af' = af$ ✓

b) 3×3 matrices with determinant 1 Not a subspace

$\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$ does not have det. 1

$\begin{bmatrix} 1&&0 \\ &1& \\ 0&&1 \end{bmatrix} + \begin{bmatrix} 1&&0 \\ &1& \\ 0&&1 \end{bmatrix} = \begin{bmatrix} 2&&0 \\ &2& \\ 0&&2 \end{bmatrix}$

c) polynomials whose degree is an even number

Not a subspace

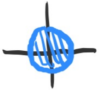
① 0 has even degree ✓

$(x^2 + x) + (-x^2) = x$

③ p has even deg, $a \cdot p$ has degree 0 or $\deg(p)$ ✓

d) vectors in $\mathbb{R}^{n\ 2}$ whose norm is at most 1

Not a subspace



① $\vec{0}$ has norm 0

② $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow$ norm is $\sqrt{2} > 1$

③ $3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \leftarrow$ norm 3

② What is the dimension of the subspace $\operatorname{span}\{1+x, x+x^2, x^2-1\}$ of $\mathbb{P}_2$? 2

$B = \{x^2, x, 1\}$ $(x^2+x)-(x+1) = x^2-1$

$[1+x]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $[x+x^2]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $[x^2-1]_B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$1+x = 0\cdot x^2 + 1\cdot x + 1\cdot 1$

$\dim\left(\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}\right) = 2$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{swap } R_1 \,\&\, R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ REF}$$

rank $= 2 \Rightarrow$ dim of the subspace is 2

③ $\vec{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

a) $T: \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation

s.t.
$$T(\vec{v}_1) = 2\vec{v}_1$$
$$T(\vec{v}_2) = \vec{v}_1 + 2\vec{v}_2$$
$$T(\vec{v}_3) = 3\vec{v}_3$$
$$T(\vec{v}_4) = \vec{v}_1$$

$$T(\vec{v}_1) = 2\vec{v}_1$$
$$= \underline{2} \cdot \vec{v}_1 + \underline{0} \cdot \vec{v}_2 + \underline{0} \cdot \vec{v}_3 + \underline{0} \cdot \vec{v}_4$$
$$[T(\vec{v}_1)]_B = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Find a basis B for $\mathbb{R}^4$ and the matrix for T relative to that basis.

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \}$

$${}_B[T]_B = \begin{bmatrix} | & | & | & | \\ [T(\vec{v}_1)]_B & [T(\vec{v}_2)]_B & [T(\vec{v}_3)]_B & [T(\vec{v}_4)]_B \\ | & | & | & | \end{bmatrix}$$

b) Find an invertible matrix P s.t. PAP^{-1} is the standard matrix for T (A is the answer to part (a), i.e. ${}_B[T]_B$)

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} \qquad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & -1 & 1 & 1 \\ 6 & 3 & 1 & 0 \\ 8 & 3 & 1 & 0 \end{bmatrix}$$

$$[T]_{std} = \underset{std \leftarrow B}{P} \; {}_B[T]_B \; \underset{B \leftarrow std}{P} = \underset{std \leftarrow B}{P}^{-1}$$

$$\underset{std \leftarrow B}{P} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_4 \\ | & & | \end{bmatrix} \qquad \underset{std \leftarrow B}{P} [\vec{v}]_B = \vec{v}$$

example $[\vec{u}]_B = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \end{bmatrix}$

$\Rightarrow \vec{u} = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 2 \cdot \vec{v}_3 + 5 \cdot \vec{v}_4$

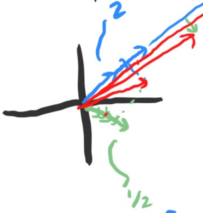
$\vec{v} \in \mathbb{R}^4$, $\quad [\vec{v}]_{std} = \vec{v}$

example: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

$$A = PDP^{-1} \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

diagonal



Compute A^{11}

$$= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})(PDP^{-1})$$

$$= PD(\underbrace{\cancel{P^{-1}P}}_{I})D(\underbrace{\cancel{P^{-1}P}}_{I})D \cdots D(\underbrace{\cancel{P^{-1}P}}_{I})DP^{-1}$$

$$= PD \cdot D \cdot D \cdots D \cdot P^{-1}$$

$$= PD^{11}P^{-1} = P\begin{bmatrix} \lambda_1^{11} & 0 \\ 0 & \lambda_2^{11} \end{bmatrix}P^{-1}$$

eigenvalues for A

eigenvectors for A

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 1\cdot(-2) & 1\cdot 1 + 4\cdot 1 \\ 1\cdot(-2) + (-2)\cdot(4) & 1\cdot(-2) + 4\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}$$

$$\underbrace{A \cdot A \cdot A \cdot A \cdots A}_{11 \text{ times}}$$

~~$A^2 \quad A^4 \quad A^8 \quad A^{10} \quad A^{11}$~~

MT2 practice 1

(6) W is a subspace of $\mathbb{R}^n$

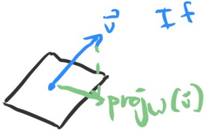
$P = [\text{proj}_W]_{std}$

a) Show $P^2 = P$

If $\vec{v} \in \mathbb{R}^n$, $P^2\vec{v} = \text{proj}_W(\text{proj}_W(\vec{v}))$ in W

$= \text{proj}_W(\vec{v})$ b/c proj. onto W of a vector in W is the vector itself

$= P\vec{v}$



Since $P^2\vec{v} = P\vec{v}$ for all vectors in $\mathbb{R}^n$, $P^2 = P$

b) Show eigenvalues of P are all 0 or 1

Suppose λ is an eigenvalue of P with eigenvector $\vec{v}$.

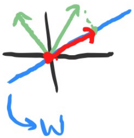
$P^2\vec{v} = P(P\vec{v}) = P(\lambda\vec{v}) = \lambda(P\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$

$P^2\vec{v} = P\vec{v} = \lambda\vec{v}$ $\Rightarrow$ $\lambda\vec{v} = \lambda^2\vec{v}$ $\Rightarrow$ $\lambda = \lambda^2$ (b/c $\vec{v} \neq \vec{0}$)

$\Rightarrow$ $\lambda = 0$ or 1

$\mathbb{R}^2$

c) What is $E_1 = W$ $E_0 = W^\perp$



$E_1 = \{\vec{v} \in \mathbb{R}^n \mid P\vec{v} = 1 \cdot \vec{v}\}$

(1) contains every vector in W

(2) If $\text{proj}_W(\vec{v}) = \vec{v}$ $\Rightarrow$ $\vec{v} \in W$ (b/c proj_W always gives something in W)

Row(A), Col(A), Null(A^T), Col(A^T), Col$(A)^\perp$ | A is $n \times m$

① rows of A = cols of A^T

cols of A = rows of A^T

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

② $A\vec{v}$ = entries are dot products of $\vec{v}$ with each row of A

③ $\vec{v}$ orthog. to $\vec{u}_1, \dots, \vec{u}_k \Rightarrow \vec{v} \in \text{span}\{\vec{u}_1, \dots, \vec{u}_k\}^\perp$

dim(Col(A)) = dim(Row(A))

① Row(A) = Col(A^T)

② Col(A) = Row(A^T)

③ Col$(A)^\perp$ = Null(A^T)

④ Row$(A)^\perp$ = Null(A)

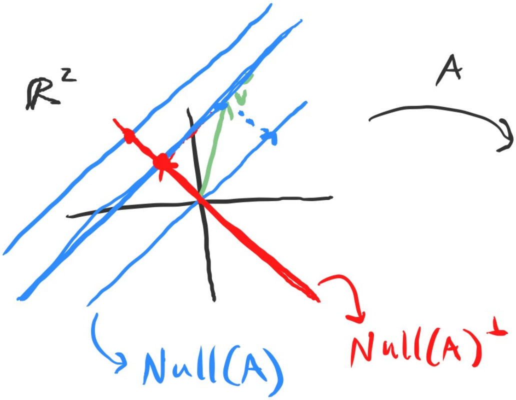
④ Rank-Nullity thm:

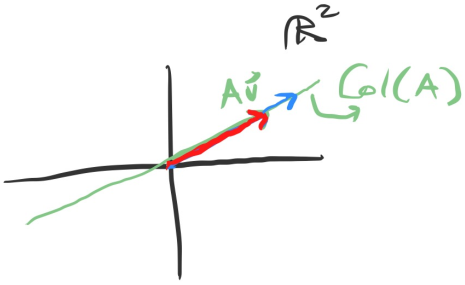
$$\dim(\text{Col}(A)) + \dim(\text{Null}(A)) = m$$

⑤ W subspace of $\mathbb{R}^n$

$$\dim(W) + \dim(W^\perp) = n$$

$$\left.\begin{aligned} &\dim(\text{Col}(A)) + \dim(\text{Col}(A)^\perp) = n \\ &\dim(\text{Col}(A)) + (\dim(\text{Null}(A^T))) = n \\ &\dim(\text{Col}(A)) + (n - \dim(\text{Col}(A^T))) = n \end{aligned}\right\} \Rightarrow \dim(\text{Col}(A)) = \dim(\text{Col}(A^T))$$





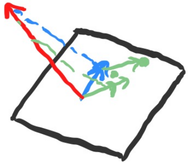
$$\text{Null}(A)^\perp = \left(\text{Row}(A)^\perp\right)^\perp = \text{Row}(A)$$

$$\vec{v} = \text{proj}_{\text{Null}(A)}(\vec{v}) + \text{proj}_{\text{Null}(A)^\perp}(\vec{v})$$

$$A\vec{v} = A\,\text{proj}_{\text{Null}(A)}(\vec{v}) + A\,\text{proj}_{\text{Null}(A)^\perp}(\vec{v})$$
$$= \vec{0} + A\,\text{proj}_{\text{Null}(A^T)}(\vec{v})$$

$\vec{v}$ & $\text{proj}_{\text{Null}(A)^\perp}(\vec{v})$ are mapped to the same vector!

(6) $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$



W

$W = \text{span}\{\vec{u}, \vec{v}\}$

Find $\text{proj}_W(\vec{x})$

student's proposed answer:

$$\text{proj}_W(\vec{x}) = \frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} + \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{7}{14}\vec{u} + \frac{3}{3}\vec{v}$$

$$= \begin{bmatrix} 3/2 \\ 0 \\ 5/2 \end{bmatrix}$$

What is wrong w/ this solution?

How can it be fixed?

$\vec{u}\cdot\vec{v} = 1-2+3 = 2 \neq 0$ so $\vec{u}$ and $\vec{v}$ are not orthogonal so we can't use them in this formula

Solution: first run Gram-Schmidt on $\vec{u}, \vec{v}$ to find an orthogonal basis