PM #2 ① i) T/F: Every invertible matrix can be written as a product of elementary matrices.

True.

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\begin{bmatrix}
r_2 & \rightarrow \\
r_2 + 5r_1 & \\
\end{bmatrix}
\begin{bmatrix}
a & b \\
c + 5a & d + 5b \\
\end{bmatrix}
\]

Matrix s.t. multiplying by it performs some elementary row operation.

\[
\begin{bmatrix}
1 & 0 \\
5 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} =
\begin{bmatrix}
a & b \\
5a + c & 5b + d \\
\end{bmatrix}
\]

Elementary matrix

\[
\begin{bmatrix}
A & I_n \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
I_n & A^{-1} \\
\end{bmatrix}
\]

Row reduce

\[
A^{-1} = B_k \ldots B_2 B_1 I_n = B_k \ldots B_2 B_1
\]

\[
\begin{bmatrix}
5 & 0 \\
0 & 2 \\
\end{bmatrix}
\]
If 

\[ V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 \geq 0 \right\} \]

is a subspace of \( \mathbb{R}^4 \) \( \text{False} \)

3 checks

1. \( 0 \in V \) ? Yes \( 0 + 0 + 0 + 0 \geq 0 \)

2. Suppose \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in V \) \( \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix} \in V \)?

Yes \( (x_1 + y_1) + \ldots + (x_4 + y_4) = (x_1 + \ldots + x_4) + (y_1 + \ldots + y_4) \geq 0 \)

3. Suppose \( \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \in V \) \( c \in \mathbb{R} \) \( \begin{bmatrix} cx_1 \\ cx_4 \end{bmatrix} \in V \) ?

No. Counterexample \( \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \in V \) but \( \begin{bmatrix} cx_1 \\ cx_4 \end{bmatrix} \notin V. \)
Find $A, B \in \mathbb{R}^{2\times2}$ s.t. $AB$ invertible but $BA$ is not invertible, or explain why no such matrices exist.

No such matrices exist,

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
Suppose $A, B \in \mathbb{R}^{2 \times 2}$ s.t. $AB$ invertible.

We will show $BA$ also invertible.

$(AB)x = 0$ has only the trivial soln $x = 0$.

$Bx = 0$ has only the trivial soln $x = 0 \Rightarrow B$ invertible (b/c $B$ square).

If not, let $\tilde{v} \neq 0$ s.t. $B\tilde{v} = 0$.

Then $(AB)\tilde{v} = A(B\tilde{v}) = A0 = 0$.

So $\tilde{v}$ is a nontrivial soln to $(AB)x = 0$.

$(AB)x = b$ has a solution for every $b \in \mathbb{R}^2$.

$Ax = \tilde{b}$ has a solution for every $\tilde{b} \in \mathbb{R}^2 \Rightarrow A$ is invertible.

(For $b \in \mathbb{R}^2$, let $\tilde{v}$ be a soln to $(AB)x = b$.

So $B\tilde{v}$ is a solution to $Ax = \tilde{b}$ b/c $A(B\tilde{v}) = (AB)v = b$.}

still true even if $A, B$ not square.
$(AB)x = \vec{0}$ has only the trivial sol'n

$\Rightarrow A^T \vec{x} = \vec{0}$ has only the trivial sol'n

This is not true if $A, B$ are not square

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

$ABx \neq \vec{0}$
Subspaces

\( \mathbb{R}^2 \)

- 0-dim subspace of \( \mathbb{R}^2 \)

- 1 dim. subspace of \( \mathbb{R}^2 \)

- 2 dim subspace of \( \mathbb{R}^2 \)

\( \rightarrow \) 2 dim. subspace of \( \mathbb{R}^3 \) (behaves "just like" \( \mathbb{R}^2 \))
Definition of a subspace of $\mathbb{R}^n$:
A set $V \subseteq \mathbb{R}^n$ is a subspace if all of the following hold:

1. $\mathbf{0} \in V$
2. For any $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} \in V$
3. For any $\mathbf{x} \in V$ and $c \in \mathbb{R}$, $c \cdot \mathbf{x} \in V$
Rank of $A$:

dimension of $\text{Col}(A) = \text{number of lin. ind. vectors that span } \text{Col}(A)$

$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ 3 & 2 & 5 \end{bmatrix}$

$\text{Col}(A)$ is spanned by

$\left[ \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{3}{5} \end{bmatrix} \right] = \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$

$\text{rank}(A) = 2$

Algorithm to find $\text{rank}(A)$: Put $A$ into REF & $\text{rank} = \# \text{ of pivots}$
Suppose \( V = \text{span} \{ \vec{v}_1, \ldots, \vec{v}_n \} \) with \( \vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m \).

How to find a basis for \( V \)?

\( V \) is a subspace of \( \mathbb{R}^m \).

A basis for \( V \) is a set of vectors that span \( V \) and are linearly independent.

\[ V = \text{span}\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \} \]

basis for \( V \): \( \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \} \)
\( \{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} \} \)
\( \{ \begin{bmatrix} 3 \\ 6 \end{bmatrix} \} \)
How to find a basis for $\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\} \\
\text{Col}\begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n \end{bmatrix} \\
\text{A} \\
\text{algorithm: row reduce A to put in in REF } \& \text{ take columns of A corresponding to pivot columns in the REF matrix} \\
\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\
\{ \begin{bmatrix} 2 \end{bmatrix} \} \text{ is a basis}
A, B matrices such that AB invertible.

\[(AB)x = \vec{0}\] has a unique sol'n

\[\Rightarrow Bx = \vec{0}\] has a unique sol'n

Suppose \(Bx = \vec{0}\) has a nontrivial solution. This means there is \(\vec{v} \neq \vec{0}\) s.t. \(B\vec{v} = \vec{0}\).

Therefore \((AB)\vec{v} = A(B\vec{v}) = A\vec{0} = \vec{0}\)

\[\Rightarrow \vec{v} \text{ is a nontrivial sol'n to } (AB)x = \vec{0}\]

\[B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\]

\[BA \text{ is invertible}\]
If $AB$ invertible $\iff$ 
$\text{Col}(B) \cap \text{Null}(A) = \{0\}$
& $\text{Null}(B) = \{0\}$
& $AB$ square

$AB\bar{v} = 0 \Rightarrow B\bar{v} \in \text{Null}(A)$
$\Rightarrow B\bar{v} = 0$
$\Rightarrow \bar{v} \in \text{Null}(B)$
$\Rightarrow \bar{v} = 0$
PM #1 (a) **TLP:** If $A$ is $m \times n$, $\vec{b} \in \mathbb{R}^m$ the set of solutions to $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$ **False**

3 checks:

1. **Contains $\vec{0}$?**

   $A \cdot \vec{0} = \vec{0}$ not equal to $\vec{0}$ if $\vec{b} \neq \vec{0}$

   So $\vec{0}$ is not a solution to $A\vec{x} = \vec{b}$ if $\vec{b}$ is nonzero.

   
   

   **Example:**

   $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  \hspace{1cm} $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  \hspace{1cm} $\vec{0}$ is not in the set of solutions of $A\vec{x} = \vec{b}$
A, B \text{n}\times\text{n}, B \text{ is invertible}, AB \text{ is invertible}

Show A \text{ is invertible.}

Because A is square, it's enough to show that \(A\vec{x} = \vec{b}\) has a solution for every \(\vec{b} \in \mathbb{R}^n\).

Let \(\vec{b} \in \mathbb{R}^n\). Since AB invertible, \((AB)\vec{x} = \vec{b}\) has a solution. Let \(\vec{\nu}\) be a solution.

\[
A (B\vec{\nu}) = (AB)\vec{\nu} = \vec{b}
\]

\(\Rightarrow B\vec{\nu}\) is a solution to \(A\vec{x} = \vec{b}\).

If A, B not square

\(AB\) invertible \(\Rightarrow A\vec{x} = \vec{0}\) has only the trivial sol'n
If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation, \( T \) is invertible if \( T \) is 1-to-1 & onto. (\( T \) invertible \( \Rightarrow n = m \))

If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \), the inverse of \( T \), \( T^{-1} \), is a linear transformation \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) s.t.

\[
T \circ T^{-1} = \text{id}_{\mathbb{R}^n}
\]

\[
T^{-1} \circ T = \text{id}_{\mathbb{R}^n}
\]

To calculate, use the standard matrix of \( T \).

\[
[T^{-1}]_{\text{std}} = [T]_{\text{std}}^{-1}
\]

To find \( A^{-1} \), \([A \ | \ I_n]\) row reduce \([I_n \ | A^{-1}]\)
$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotation by $90^\circ$ counterclockwise

$T^{-1}$ should be rotation by $90^\circ$ clockwise

$[T]_{\text{std}} = \left[ \begin{array}{cc} T((0,0)) & T((0,1)) \end{array} \right] = \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right]$

$\left[ \begin{array}{ccc} 0 & -1 & 1 \\ 1 & 0 & 0 \end{array} \right]$ swap $R_1$ & $R_2$

$\left[ \begin{array}{ccc} 0 & -1 & 1 \\ 1 & 0 & 0 \end{array} \right] R_2 = -R_2 \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right]$

$[T^{-1}]_{\text{std}} = [T]_{\text{std}}^{-1} = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]$
Col(A) = span of the columns of A
Null(A) = set of solutions to $A\vec{x} = \vec{0}$

basis of a subspace is a set of linearly independent vectors in the subspace that span the whole subspace.

dimension of a subspace = size of any basis
(all bases for a subspace have the same size)

$\dim(\text{Col}(A)) = \text{rank}(A) = \# \text{ of pivots when } A \text{ is put in REF}$

to find a basis: put $A$ in REF & use columns of $A$ corresponding to pivot columns of REF matrix

$\dim(\text{Null}(A)) = \# \text{ of free variables when } A \text{ is put in REF}$
to find a basis: write solutions to $A\vec{x} = \vec{0}$ in parametric form
\[ A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix} \]

\[ \dim(\text{Null}(A)) = 1 \]

\[ \text{rank}(A) = 2 \]

A basis for \( \text{Col}(A) \) is \( \left\{ \begin{bmatrix} 1 \\ 1/3 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\} \)

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 2 & 5
\end{bmatrix} \\ R_2 \leftarrow R_2 - 2R_1 \quad R_3 \leftarrow R_3 - 3R_1 \\
\begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 2 \\
0 & 0 & 0
\end{bmatrix} \quad RREF
\]

\[ \begin{aligned}
\mathbf{x}_1 &= -\mathbf{x}_3 t \\
\mathbf{x}_2 &= -\mathbf{x}_3 t \\
\mathbf{x}_3 &= \text{free} = t
\end{aligned} \]

A basis for \( \text{Null}(A) \) is \( \left\{ \begin{bmatrix} -1 \end{bmatrix} \right\} \)

\[ s \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \text{A basis for } \text{Null}(A) \text{ would be } \left\{ \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} \right\} \]