$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}$$

What is Col(A)?

Basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

Col(A) = span of columns of A

Basis for subspace is a set of vectors which are lin. independent & span the entire subspace

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\text{lin. ind.}}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Algorithm: remove columns that become free variables in REF

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

free variable

REF

2.9 27 $\vec{b}_1, \dots \vec{b}_p$ span W

$\vec{a}_1, \dots \vec{a}_q$ in W $q > p$

Prove $\vec{a}_1 \dots \vec{a}_q$ lin. dep.

Lin. dep. means there are constants $c_1 \dots c_q \in \mathbb{R}$, not all 0, such that $c_1 \cdot \vec{a}_1 + \dots + c_q \cdot \vec{a}_q = \vec{0}$

Algorithm: write vectors as columns of a matrix, row reduce, & check for free variables

$$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_q \\ | & & | \end{bmatrix} \qquad B = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix}$$

a) Explain why for each $\vec{a}_i$ there is a vector $\vec{c}_i$ such that $\vec{a}_i = B\vec{c}_i$

$\vec{a}_i \in \text{span}\{\vec{b}_1, \dots, \vec{b}_p\}$ $\quad \vec{a}_i = \underbrace{r_1 \cdot \vec{b}_1 + \dots + r_p \cdot \vec{b}_p}_{B \begin{bmatrix} r_1 \\ \vdots \\ r_p \end{bmatrix} \}p}$

b) $C = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_q \\ | & & | \end{bmatrix}$

Find $\vec{u} \neq \vec{0}$ s.t. $C\vec{u} = \vec{0}$

C is a $p \times q$ matrix $p < q$

$\Rightarrow C$ has at least 1 free variable in REF

c) Show $A\vec{u} = \vec{0}$ $\quad A = BC$ (from (a))

$$\begin{aligned} A\vec{u} &= (BC)\vec{u} \\ &= B(C\vec{u}) \\ &= B \cdot \vec{0} \\ &= \vec{0} \end{aligned}$$

2.3 12c T/F If $A\vec{x}=\vec{b}$ has at least 1 sol'n for each $\vec{b} \in \mathbb{R}^n$ then the solution is unique for each $\vec{b}$

A is $n \times n$

pivot in every row
$\Rightarrow$ n pivots
$\Rightarrow$ pivot in every column.

True (only true because A is square).

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \end{array}\right]$$

infinitely many sol'ns for every a, b

free variable

3.2 (27a) T/F: A row replacement operation does not affect the determinant of a matrix

True. A is 2×2

add k times R_1 to R_2 $\quad k \in \mathbb{R}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ k \end{bmatrix}$ $\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 = R_2 + k\cdot R_1} \begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad ad-bc$

$$\begin{aligned} & a(d+kb) - b(c+ka) \\ &= ad + \cancel{akb} - bc - \cancel{bka} \\ &= ad - bc \end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix}$$

$= 1$

$$\det \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix} = \det\left(\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\right) \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

(11) A is an $n \times m$ matrix. How many sol'ns does $A\vec{x} = \vec{b}$ have if

a) $\text{Null}(A) = \{\vec{0}\}$ 1 sol'n.

$\vec{b} \in \text{Col}(A)$

solutions to $A\vec{x} = \vec{0}$ ($A\vec{x} = \vec{0}$ has only the trivial sol'n $\Rightarrow$ pivot in every colum)

$\text{Col}(A) =$ span of columns of A

($\vec{b} \in \text{Col}(A) \Rightarrow A\vec{x} = \vec{b}$ has a sol'n)

b) $\text{Null}(A) \neq \{\vec{0}\}$ $\Rightarrow$ free variable

$b \in \text{Col}(A)$ $\Rightarrow$ there is a sol'n

Infinitely many

$A \cdot \vec{x}$ is always a lin. combination of the columns of A

c) $b \notin \text{Col}(A)$

0 sol'ns

(12) If A is an $n \times n$ invertible matrix, what are $\text{Null}(A)$ and $\text{Col}(A)$?

$\text{Null}(A) = \{\vec{0}\}$

$A\vec{x} = \vec{0}$ has only the trivial sol'n

$$A\vec{x} = \vec{0} \qquad A^{-1}(A\vec{x}) = A^{-1}\vec{0}$$

$$A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} = I_n\vec{x} = \vec{x}, \qquad A^{-1}\vec{0} = \vec{0}, \qquad \vec{x} = \vec{0}$$

$\text{Col}(A) = \mathbb{R}^n$

$\vec{b} \in \mathbb{R}^n$. Is $\vec{b} \in \text{Col}(A)$

$\Leftrightarrow$ Does $A\vec{x} = \vec{b}$ have sol'n

Yes. $A^{-1}\vec{b}$ is a sol'n

A is invertible $\Rightarrow$ pivot in every row & col.

$\Rightarrow$ $A\vec{x} = \vec{b}$ always has a sol'n (for all $\vec{b}$)

$\Rightarrow$ every $\vec{b} \in \mathbb{R}^n$ is a lin. combination of columns of A

Suppose A, C are invertible

$$AB = C$$

Is B invertible? Yes.

A $n \times n$

B $n \times n$

C $n \times n$

If so $(AB)^{-1} = B^{-1}A^{-1}$

$$B^{-1}A^{-1} = C^{-1}$$

$$B^{-1} = C^{-1}A$$

Because B is square, enough to check $B\vec{x} = \vec{0}$ has a unique sol'n.
Suppose $B\vec{v} = \vec{0}$. Then
$\vec{0} = AB\vec{v} = C\vec{v} \Rightarrow \vec{v} = \vec{0}$ b/c C invertib

Try to check if $C^{-1}A$ is an inverse of B

$$(C^{-1}A)B = C^{-1}(AB) = C^{-1}C = I_n \checkmark$$

If a matrix is square & has a left inverse then it's invertible

(1) $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ $m \times n$ matrix

$$\vec{v}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \cdots \quad \vec{v}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$T: \mathbb{R}^n \to \mathbb{R}^m \qquad T(\vec{x}) = A\vec{x}$$

REF/pivots	systems of lin. eqns	matrix eq'ns	span/lin. dep. of vectors	linear transformations
$[A \mid b]$ has no pivot in last column in REF	$a_{11}x_1 + \ldots + a_{1n}x_n = b_1$ $\vdots$ $a_{m1}x_1 + \ldots a_{mn}x_n = b_m$ has a sol'n	$A\vec{x} = \vec{b}$ has a sol'n	$\vec{b} \in \operatorname{span}\{\vec{v}_1, \ldots \vec{v}_n\}$	$\vec{b} \in \operatorname{range}(T)$
pivot in every column of $[A \mid b]$ except for last column	$a_{11}x_1 + \ldots = b_1$ $\vdots$ $a_{m1}x_1 \ldots = b_n$ has a unique sol'n	$A\vec{x} = \vec{b}$ has a unique sol'n	$\vec{b}$ can be written as a lin. comb. of $\vec{v}_1, \ldots, \vec{v}_n$ in exactly 1 way	there is exactly 1 $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$

REF/pivots	lin. eq'ns	matrix eq'ns	vectors	lin. transformation
pivot in every col. of A (after you put it in REF)	$a_{11}x_1 + \cdots = 0$ $\vdots$ $a_{m1} \cdots = 0$ has a unique sol'n	$A\vec{x} = \vec{0}$ has a unique sol'n	$\vec{v}_1 \dots \vec{v}_n$ lin. ind. $\mathrm{Null}(A) = \{\vec{0}\}$	T is 1-to-1
pivot in every row of A in REF	$a_{11} \cdots = c_1$ $\vdots$ $\cdots = c_m$ has a sol'n for all $c_1 \dots c_m$	$A\vec{x} = \vec{c}$ has a sol'n for all $\vec{c} \in \mathbb{R}^m$	$\mathrm{span}\{\vec{v}_1, \dots \vec{v}_n\} = \mathbb{R}^m$ $\Updownarrow$ $\mathrm{Col}(A) = \mathbb{R}^m$	T is onto

"has a sol'n" = "consistent"

Wed. 8 pm PST

Due Fri. 8 am PST

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

4×2

columns are vectors in $\mathbb{R}^4$

2 of them