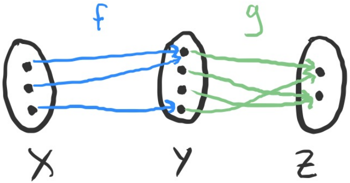
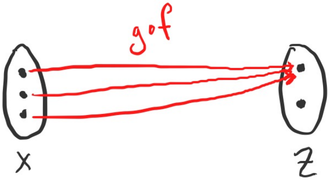
Composition of Functions

If $f : X \to Y$ and $g : Y \to Z$ are functions then the composition of g and f is the function $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g(f(x))$$

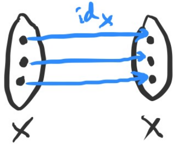




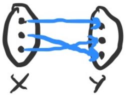
Inverses of Functions

1. For any set X, the identity function on X is the function $id_X : X \to X$ defined by

$$id_X(x) = x$$



2. A function $f : X \to Y$ is called invertible if it is both one-to-one and onto.

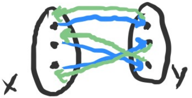


3. If $f : X \to Y$ is invertible then there is a function $f^{-1} : Y \to X$, called the inverse of f such that

$$f \circ f^{-1} = id_Y$$

and

$$f^{-1} \circ f = id_X$$



"follow the arrows backwards"

① $f: \mathbb{N} \to \mathbb{N}$ $g: \mathbb{N} \to \{0,1\}$

$f(n) = 2n$ $g(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

a) $(g \circ f)(5) =$ $g(f(5)) = g(10) = 0$

b) range of $g \circ f$? $\text{range}(g \circ f) = \{0\}$

1 is <u>not</u> in the range because for every $n \in \mathbb{N}$, $f(n)$ is even so $g(f(n)) = 0$

c) Is $g \circ f$ 1-to-1? No. $(g \circ f)(1) = (g \circ f)(2) = 0$

Is $g \circ f$ onto? No. Codomain is $\{0,1\}$, but 1 is not in the range

(2) Check if each function is invertible. If so, find its inverse.

a) $f: \{1, 2, 3\} \to \{4, 5, 6\}$

$1 \mapsto 5$
$2 \mapsto 6$
$3 \mapsto 4$

It is invertible.

Inverse: $f^{-1}: \{4, 5, 6\} \to \{1, 2, 3\}$

$f^{-1}(4) = 3 \quad f^{-1}(5) = 1 \quad f^{-1}(6) = 2$

b) $g: \mathbb{R} \to \mathbb{R} \quad g(x) = x^2$ Not invertible.

Not 1-to-1: $g(-2) = g(2) = 4$

Not onto: -2 is not in the range

c) $h: \mathbb{R} \to \mathbb{R} \quad h(x) = 2x - 3$ It is invertible.

Inverse: $h^{-1}: \mathbb{R} \to \mathbb{R} \quad h^{-1}(x) = \frac{x+3}{2}$

Check: $h^{-1}(h(x)) = \frac{(2x-3)+3}{2} = x$

$h(h^{-1}(x)) = 2\left(\frac{x+3}{2}\right) - 3 = x$

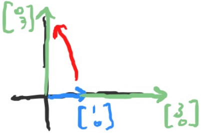
Linear Transformations	Matrices
$S: \mathbb{R}^n \to \mathbb{R}^m$	$[S]_{std} = A$
$T: \mathbb{R}^k \to \mathbb{R}^\ell$	$[T]_{std} = B$
$S(\vec{v})$	$A \cdot \vec{v}$
$S \circ T$ (only makes sense if $\ell = n$)	$[S \circ T]_{std} = A \cdot B$
S is 1-to-1	A has a pivot in every col. in REF
S is onto	A has a pivot in every row in REF
S^{-1}	$[S^{-1}]_{std} = A^{-1}$
$id_{\mathbb{R}^n}$	I_n

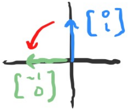
① For each item, find $[S \circ T]_{std}$

a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$ counterclockwise

$T : \mathbb{R}^2 \to \mathbb{R}^2$ expansion by a factor of 3 horizontally

$$[S \circ T]_{std} = \begin{bmatrix} | & | \\ S \circ T(\vec{e}_1) & S \circ T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$$





Alternate method: $[S \circ T]_{std} = [S]_{std} \cdot [T]_{std}$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$$

b) $S : \mathbb{R}^2 \to \mathbb{R}^2$ expansion by 3 horizontally

$T : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$ counterclockwise

$$[S \circ T]_{std} = [S]_{std}[T]_{std} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$$

Notice that the result is different from part (a). Order matters.

c) $S : \mathbb{R} \to \mathbb{R}^3$ $\qquad S(x) = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$

$T : \mathbb{R}^3 \to \mathbb{R}$ $\qquad T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x+y+z$

$$[S \circ T]_{std} = \begin{bmatrix} | & | & | \\ S\circ T(\vec{e}_1) & S\circ T(\vec{e}_2) & S\circ T(\vec{e}_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ S(1) & S(1) & S(1) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Alternately: $[S]_{std} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $[T]_{std} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

d) $S: \mathbb{R}^2 \to \mathbb{R}^4$ has std. matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 1 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$ counterclockwise

$$[S \circ T]_{std} = [S]_{std} [T]_{std}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & -2 \\ 0 & -3 \\ 1 & -4 \end{bmatrix}$$

(2) Check whether $S \circ T$ is invertible. If so, find the standard matrix of its inverse.

a) $S: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$ counterclockwise
$T: \mathbb{R}^2 \to \mathbb{R}^2$ expansion by 3 horizontally

From problem 1a : $[S \circ T]_{std} = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \xrightarrow{\text{swap } R_1 \& R_2} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ REF

pivot in every row $\Rightarrow$ onto
pivot in every col. $\Rightarrow$ 1-to-1

$S \circ T$ is invertible

$$[(S \circ T)^{-1}]_{std} = [S \circ T]_{std}^{-1}$$

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_1 \& R_2} \left[\begin{array}{cc|cc} 3 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 = \frac{1}{3}R_1} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1/3 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 = -R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1/3 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

I_2 $\qquad$ $[S \circ T]_{std}^{-1}$

Standard matrix of $(S \circ T)^{-1}$ is : $\begin{bmatrix} 0 & 1/3 \\ -1 & 0 \end{bmatrix}$

b) $S: \mathbb{R}^2 \to \mathbb{R}^2$ expansion by 3 horizontally

$T: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$ counterclockwise

From problem 1b : $[S \circ T]_{std} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{swap } R_1 \& R_2} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \text{ REF}$$

pivot in every row $\Rightarrow$ onto

pivot in every col. $\Rightarrow$ 1-to-1

$S \circ T$ is invertible

$$\left[\begin{array}{cc|cc} 0 & -3 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \& R_2]{\text{swap}} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 0 \end{array}\right]$$

$\downarrow I_2$ $\quad \searrow [S \circ T]_{std}^{-1}$

$$[(S \circ T)^{-1}]_{std} = [S \circ T]_{std}^{-1} = \begin{bmatrix} 0 & 1 \\ -1/3 & 0 \end{bmatrix}$$

Method 2 :

$$[S]_{std}^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[T]_{std}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$[S \circ T]_{std}^{-1} = ([S]_{std}[T]_{std})^{-1}$$

$$= [T]_{std}^{-1}[S]_{std}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1/3 & 0 \end{bmatrix}$$

2c $S: \mathbb{R} \to \mathbb{R}^3$ $S(x) = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$ $T: \mathbb{R}^3 \to \mathbb{R}$ $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x+y+z$

From problem 1c: $[S \circ T]_{std} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{RREF}$$

No pivot in 2^{nd} & 3^{rd} rows $\Rightarrow$ not onto

No pivot in 2^{nd} & 3^{rd} col.s $\Rightarrow$ not 1-to-1

$S \circ T$ is not invertible.

2d $S : \mathbb{R}^2 \to \mathbb{R}^4$ has standard matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 1 \end{bmatrix}$

$T : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$ counterclockwise

From problem 1d: $[S \circ T]_{std} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \\ 0 & -3 \\ 1 & -4 \end{bmatrix}$

$$\begin{bmatrix} 0 & -1 \\ 1 & -2 \\ 0 & -3 \\ 1 & -4 \end{bmatrix} \xrightarrow[\& R_2]{\text{swap } R_1} \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 0 & -3 \\ 1 & -4 \end{bmatrix} \xrightarrow{R_4 = R_4 - R_1} \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 0 & -3 \\ 0 & -2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_2} \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\xrightarrow{R_4 = R_4 - 2R_2} \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ REF

No pivot in 3rd or 4th rows
$\Rightarrow$ not onto

$S \circ T$ is not invertible.

More general fact: A linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ can only be invertible if $n = m$.