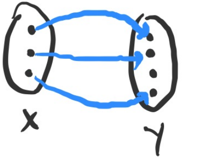
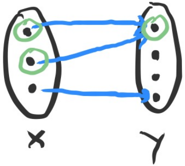
$f: X \longrightarrow Y$ is one-to-one if there are no $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$



one-to-one



not 1-to-1

$\{0, 1, 2, 3, \ldots\}$

(1) a) $f: \mathbb{N} \to \mathbb{N}$ $\quad f(n) = 2n$ $\quad$ 1-to-1 $\quad 2n = 2m \Rightarrow n = m$

not onto $\quad$ 1 is not in the range

b) $g: \mathbb{N} \to \{0, 1\}$ $\quad g(n) = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$ $\quad g(0) = 0$, $g(1) = 1$

not 1-to-1 $\quad g(2) = g(4) = 0$

onto

c) $h : \{1,2,3\} \to \{4,5,6\}$

$1 \mapsto 5$

$2 \mapsto 6$

$3 \mapsto 5$

not 1-to-1

$h(1) = h(3) = 5$

not onto

4 is not in the range

d) $k : \mathbb{Z} \to \mathbb{R}^2$ $\quad k(n) = \begin{bmatrix} n \\ -n \end{bmatrix}$

one-to-one

not onto

$\begin{bmatrix} n \\ -n \end{bmatrix} = \begin{bmatrix} m \\ -m \end{bmatrix} \Rightarrow n = m$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ not in the range $\quad \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$ also not in the range

e) $j : \{1, 2, 3, \dots\} \to \mathbb{N}$ $\quad j(n) = n^{th}$ digit of π $\quad 3.141$

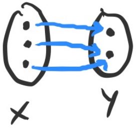
not 1-to-1

$j(2) = j(4) = 1$

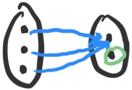
not onto

10 is not in the range

$f : X \to Y$ onto if for each $y \in Y$ there is some $x \in X$ such that $f(x) = y$



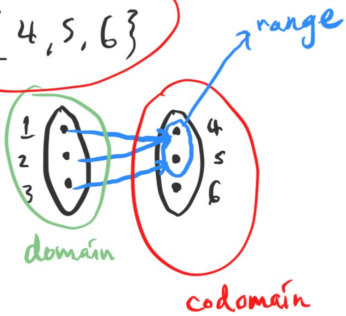
onto



not onto

$$f : \{1, 2, 3\} \to \{4, 5, 6\}$$

$$f(1) = 4$$
$$f(2) = 4$$
$$f(3) = 5$$



a) lin. transformation 1-to-1 & onto. $\mathbb{R}^n \to \mathbb{R}^m$ $n = m$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(\vec{v}) = \vec{v}$

identity map

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

I_2

one-to-one:

$T(\vec{v}) = T(\vec{u}) \Rightarrow \vec{v} = \vec{u}$

onto: for any $\vec{v}$

$T(\vec{v}) = \vec{v}$ so $\vec{v}$ is in range(T)

REF

pivot in every col. $\Rightarrow$ 1-to-1

pivot in every row $\Rightarrow$ onto

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

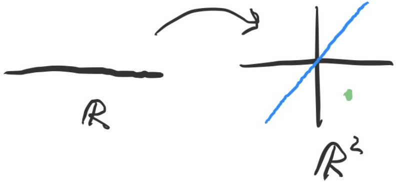
$\mathbb{R}^2 \to \mathbb{R}^2$

rotation by 90°

etc.

b) 1-to-1, not onto $\mathbb{R}^n \to \mathbb{R}^m$ $n < m$

$T: \mathbb{R} \to \mathbb{R}^2$



$$T(x) = \begin{bmatrix} x \\ x \end{bmatrix}$$

1-to-1 because $\begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} \Rightarrow x = y$

not onto. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ not in the range

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = [T]_{std}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

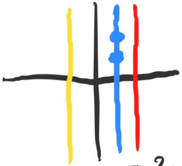
pivot in every col. $\Rightarrow$ 1-to-1

no pivot in 2^{nd} row $\Rightarrow$ not onto

c) onto, not 1-to-1 $\quad \mathbb{R}^n \to \mathbb{R}^m \quad n > m$

$T: \mathbb{R}^2 \to \mathbb{R}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x$



$\mathbb{R}^2$

$\mathbb{R}$

not one-to-one: $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 1$

onto: for each $x \in \mathbb{R}$, $T\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) = x$

so $x \in \text{range}(T)$

$[T]_{std} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ REF

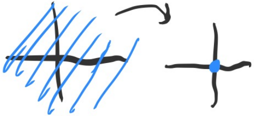
no pivot in 2^{nd} column $\Rightarrow$ not 1-to-1

pivot in every row $\Rightarrow$ onto

(3d) A linear transformation that is neither one-to-one nor onto

stupid example : $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(\vec{x}) = \vec{0}$



not one-to-one $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

not onto : $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ not in the range

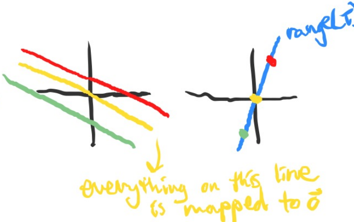
$[T]_{std} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ REF

no pivot in either col $\Rightarrow$ not 1-to-1

no pivot in either row $\Rightarrow$ not onto

2^{nd} example : $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(\vec{v}) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \vec{v}$



4 $f: \mathbb{R} \to \mathbb{R}^3$ $\quad f(x) = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$ $\quad g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x+y+z$

$g: \mathbb{R}^3 \to \mathbb{R}$

a) $g \circ f$ function from $\mathbb{R} \to \mathbb{R}$

$$g \circ f(x) = g(f(x)) = g\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}\right) = x + x + x = 3x$$

b) $f \circ g$ Function from $\mathbb{R}^3 \to \mathbb{R}^3$

$$f \circ g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = f\left(g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right) = f(x+y+z) = \begin{bmatrix} x+y+z \\ x+y+z \\ x+y+z \end{bmatrix}$$

c)

$$[g \circ f]_{std} = [g \circ f(\vec{e}_1)] = [g \circ f(1)] = [3]$$

$$[f \circ g]_{std} = \begin{bmatrix} | & | & | \\ f \circ g(\vec{e}_1) & f \circ g(\vec{e}_2) & f \circ g(e_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix}$$