Functions

1. For each function below, state whether it is one-to-one. If it is not one-to-one then find two elements of the domain which are mapped to the same element of the codomain.

   (a) $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$.

   (b) $g\colon \mathbb{N} \to \{0, 1\}$ defined by

   $$g(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

   (c) $h\colon \{1, 2, 3\} \to \{4, 5, 6\}$ defined by

   $$h(n) = \begin{cases} 5 & \text{if } n = 1 \\ 6 & \text{if } n = 2 \\ 5 & \text{if } n = 3. \end{cases}$$

   (d) $k\colon \mathbb{Z} \to \mathbb{R}^2$ defined by

   $$k(n) = \begin{bmatrix} n \\ -n \end{bmatrix}.$$

   (e) $j\colon \{1, 2, 3, \ldots\} \to \mathbb{N}$ defined by $j(n) =$ the n^{th} digit of π.

2. For each function in the previous question, state whether it is onto. If it is not onto, find an element of the codomain that is not in the range.

3. For each item below, either give an example or explain why no example exists.

   (a) A linear transformation that is one-to-one and onto.

   (b) A linear transformation that is one-to-one but not onto.

   (c) A linear transformation that is onto but not one-to-one.

   (d) A linear transformation that is neither one-to-one nor onto.

4. Let $f\colon \mathbb{R} \to \mathbb{R}^3$ be the function defined by

   $$f(x) = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$$

   and let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

   $$g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x + y + z.$$

   (a) What is $g \circ f$?

   (b) What is $f \circ g$?

   (c) Both f and g are linear transformations and therefore so are $f \circ g$ and $g \circ f$. Find the standard matrix of each.