Linear transformations

① Linear transformation is a function
$T : \mathbb{R}^n \to \mathbb{R}^m$ that preserves linear combinations: $\forall a_1 \dots a_k \in \mathbb{R}$ $\vec{v}_1 \dots \vec{v}_k \in \mathbb{R}^n$

$$T(a_1 \cdot \vec{v}_1 + \dots + a_k \cdot \vec{v}_k) = a_1 \cdot T(\vec{v}_1) + \dots + a_k \cdot T(\vec{v}_k)$$

② Linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ $\approx$ $m \times n$ matrices (of real numbers)

$$T \rightsquigarrow [T]_{std} \quad \left([T]_{std} = \begin{bmatrix} | & & | \\ T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) & \cdots & T\left(\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}\right) \\ | & & | \end{bmatrix} \right)$$

$$A \rightsquigarrow T_A \quad \left(T_A(\vec{v}) = A \cdot \vec{v} \right)$$

If you know what T does to $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ you know what it does to every vector

① $T: \mathbb{R}^2 \to \mathbb{R}^3$ linear transformation

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\3\\-1\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\\1\end{bmatrix}$$

a) $$T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = T\left(2\cdot\begin{bmatrix}1\\0\end{bmatrix} + 3\cdot\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 2\cdot T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 3\cdot T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

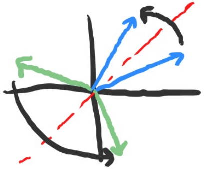
$$= 2\cdot\begin{bmatrix}5\\3\\-1\end{bmatrix} + 3\cdot\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}16\\9\\1\end{bmatrix}$$

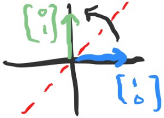
b) $$[T]_{std} = \begin{bmatrix} T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\end{bmatrix} = \begin{bmatrix}5 & 2\\3 & 1\\-1 & 1\end{bmatrix}$$

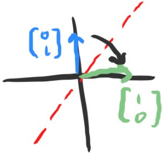
$$\begin{bmatrix}5 & 2\\3 & 1\\-1 & 1\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}5\cdot 2 + 2\cdot 3\\3\cdot 2 + 1\cdot 3\\-1\cdot 2 + 1\cdot 3\end{bmatrix} = \begin{bmatrix}16\\9\\1\end{bmatrix}$$

2a reflection across the line $x=y$



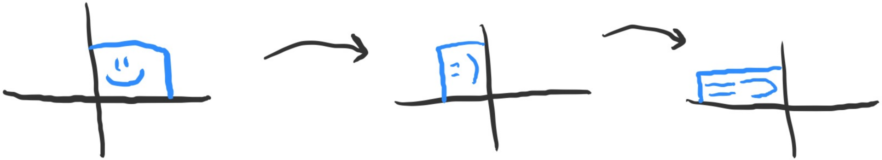
To find std matrix, check what happens to $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$

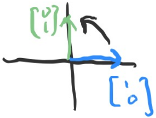


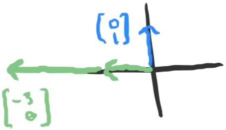


$$[T]_{std} = \begin{bmatrix} | & | \\ T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(2b) Rotation by $90°$ counterclockwise, followed by expansion by 3 in horiz. dir.







$$[T]_{std} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$$

2c Everything goes to $\vec{0}$

$$[T]_{std} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

④ $T : \mathbb{R}^2 \to \mathbb{R}^3$ defd by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \cdot y \\ y \\ x \end{bmatrix}$$

Not a linear transformation.

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$$

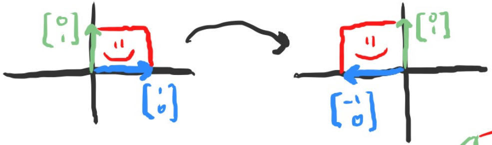
$$T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 12 \\ 4 \\ 3 \end{bmatrix} \qquad = \begin{bmatrix} 14 \\ 6 \\ 4 \end{bmatrix}$$

$$\neq$$

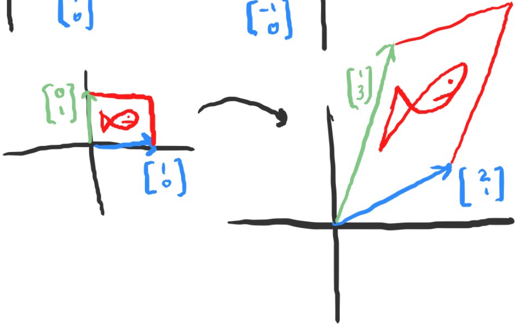
$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 24 \\ 6 \\ 4 \end{bmatrix}$$

③ Draw a picture of each linear transformation

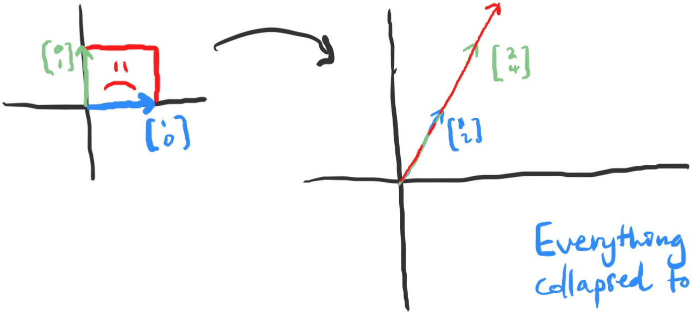
a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ Main idea: check what happens to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



b) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

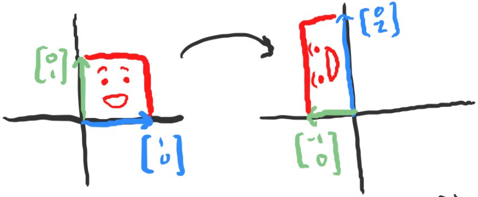


c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

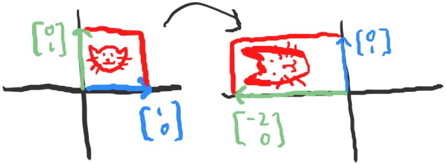


Everything is collapsed to a line

d) $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$



e) $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$



f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ everything stays exactly the same