Review \( V = \text{span} \{ 1, x, x^2 \} = \mathbb{P}_2 \)

\( T : V \to V \quad T(f) = 5 \frac{d^2 f}{dx^2} + 2 \frac{df}{dx} \)

Pick a basis for \( V \) and write the matrix for \( T \) relative to that basis

\[ B = \{ 1, x, x^2 \} \]

\[ T(1) = 0 \]

\[ T(x) = 5 \frac{d^2 x}{dx^2} + 2 \frac{dx}{dx} = 2 \]

\[ T(x^2) = 5 \frac{d^2 x^2}{dx^2} + 2 \frac{dx^2}{dx} = 5 \cdot 2 + 2 \cdot (2x) = 10 + 4x \]

\[ [T]_B = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \]
Suppose \( y_1(t) \) and \( y_2(t) \) are both sol'ns to
\[
ay'' + by' + cy = 0
\]
Show that \( c_1 y_1(t) + c_2 y_2(t) \) is also a sol'n
(where \( c_1, c_2 \in \mathbb{R} \))

\[
a (c_1 y_1(t) + c_2 y_2(t))'' + b (c_1 y_1(t) + c_2 y_2(t))' \\
+ c (c_1 y_1(t) + c_2 y_2(t))
\]

\[
= a \cdot c_1 \cdot y_1'' + a \cdot c_2 \cdot y_2'' + b \cdot c_1 y_1' + b \cdot c_2 y_2' \\
+ c \cdot c_1 y_1 + c \cdot c_2 y_2
\]

\[
= c_1 (a y_1'' + b y_1' + c y_1) + c_2 (a y_2'' + b y_2' + c y_2)
\]

\[
= c_1 \cdot 0 + c_2 \cdot 0 = 0.
\]
Finding a solution to \( ay'' + by' + cy = 0 \) = Find an element of the kernel of \( T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \) defined by \( T(f) = a \frac{d^2f}{dt^2} + b \frac{df}{dt} + cf \)

Previous exercise: Since \( T \) is a linear transformation, its kernel is a subspace—i.e., closed under addition and scalar multiplication.
Suppose $e^{2t}$ and $e^{3t}$ are both solutions to

$$ay'' + by' + cy = 0$$

and $\sin(t)$ is a solution to

$$ay'' + by' + cy = -5\cos(t) + 5\sin(t)$$

Find all solutions to

$$\sin(t) + c_1e^{2t} + c_2e^{3t}$$

Let

$$T = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c\cdot I$$

Then

$$T(\sin(t) + c_1e^{2t} + c_2e^{3t}) = T(\sin(t)) + c_1T(e^{2t}) + c_2T(e^{3t})$$

$$= (-5\cos(t) + 5\sin(t)) + c_1\cdot 0 + c_2\cdot 0$$

$$= -5\cos(t) + 5\sin(t)$$
Let $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the set of infinitely differentiable functions such that

$$T(f) = 5 \frac{d^2 f}{dt^2} + 2 \frac{df}{dt}$$

Find all eigenvectors of $T$.

$$T(f) = \lambda f$$

for some $\lambda \in \mathbb{R}$

$s f'' + 2f' = \lambda f$

$s f'' + 2f' - \lambda f = 0$

$s r^2 + 2r - \lambda$

Characteristic equation

$s r^2 + 2r - \lambda = 0$

roots: $r = \frac{-2 \pm \sqrt{4 - 4s}}{2}$

$= \frac{-1 \pm 1 + s}{s}$

$= \frac{-1 \pm \sqrt{1 + 5\lambda}}{s}$

$e^{\frac{-1 \pm \sqrt{1 + 5\lambda}}{s} t}$ is an eigenvector with eigenvalue $\lambda$. 

Solving Diff. Eqs.

1. Learn algorithms to find solutions
2. Understanding where these algorithms came from & why they work

[this is where linear algebra comes in]