Singular Value Decomposition (SVD)

$$\underset{m\times n}{A^T} \quad \underset{n\times m}{A}$$

A $n\times m$ matrix

A^TA square symmetric $m\times m$ matrix $\Rightarrow$ orth. diagonalizable

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0 \quad \lambda_{r+1}=0, \dots, \lambda_m = 0$ eigenvals of A^TA

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \dots \vec{v}_m$ orthonormal eigenvecs of A^TA

$$\frac{A\vec{v}_1}{\sqrt{\lambda_1}}, \frac{A\vec{v}_2}{\sqrt{\lambda_2}}, \dots \frac{A\vec{v}_r}{\sqrt{\lambda_r}} \longrightarrow \text{orthonormal}$$

$\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \dots$ singular values of A

$$A = \underbrace{\overbrace{\begin{bmatrix} | & & | & & | \\ u_1 & \cdots & u_r & \cdots & u_n \\ | & & | & & | \end{bmatrix}}^{n\times n}}_{=U} \underbrace{\begin{bmatrix} \sigma_1 & & 0 & 0 \\ 0 & \ddots & \sigma_r & 0 \\ & 0 & & \end{bmatrix}}_{=\Sigma,\ n\times m} \underbrace{\begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_m & - \end{bmatrix}}_{=V^T,\ m\times m}$$

$u_1 = \dfrac{A\vec{v}_1}{\sigma_1}$

complete to orth. basis

Algorithm to compute SVD A $n \times m$

① Find A^TA

② Find eigenvalues of A^TA

③ Find an orthonormal basis for $\mathbb{R}^m$ of eigenvecs of A^TA, $\vec{v}_1, \dots, \vec{v}_r, \dots \vec{v}_m$

④ Use Gram-Schmidt to complete $A\vec{v}_1, \dots A\vec{v}_r$ to an orthog. basis for $\mathbb{R}^n$ & then normalize to get an orthonormal basis

orthogonal

$$\underbrace{A\vec{v}_1, \dots A\vec{v}_r}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$ spans all of $\mathbb{R}^n$

① $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ Find SVD of A

$A^T = U \Sigma V^T$

Hint: Instead, find SVD of A^T $\Rightarrow A = V \Sigma^T U^T$

① $(A^T)^T A^T = AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

② $\det \begin{bmatrix} 2-t & 1 \\ 1 & 2-t \end{bmatrix} = (2-t)^2 - 1 = 4 - 4t + t^2 - 1$

$= t^2 - 4t + 3$

$= (t-3)(t-1)$

Eigenvalues: 3, 1

Singular values: $\sqrt{3}, \sqrt{1}$

③ Eigenspace 3: $E_3 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_1 = x_2$, x_2 free

Eigenspace 1: $E_1 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x_1 = -x_2$ x_2 free

orthonormal basis for $\mathbb{R}^2$: $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

④ $A^T \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\left(A^T \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$A^T \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\left(A^T \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

Orth. basis for $\mathbb{R}^3$: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$

orthonormal basis: $\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$$A^T = \underbrace{\begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix}}_{\Sigma \text{ same shape as } A^T} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{1} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

reduced SVD

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T = U \Sigma V^T$$

$$\begin{aligned} A &= (A^T)^T = (U \Sigma V^T)^T \\ &= (V^T)^T \Sigma^T U^T \\ &= V \Sigma^T U^T \end{aligned}$$