① Spectral Theorem

Given $T: \mathbb{R}^n \to \mathbb{R}^n$ linear transformation

Nice! There is a basis for $\mathbb{R}^n$ of eigenvectors for T

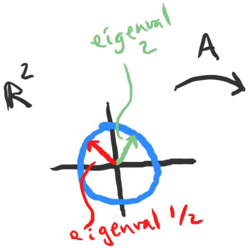
Really nice: There is an orthonormal basis for $\mathbb{R}^n$ of eigenvectors for T

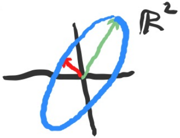
$A^T = A$

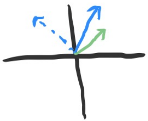
Spectral Theorem: If A is an $n \times n$ symmetric matrix whose entries are real numbers then

① There is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors for T

② All eigenvals of A are real







① Which of these are symmetric?

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Not symmetric

Not symmetric

symmetric

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

② Suppose A $n \times n$ has eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ which are orthonormal and have eigenvals $\lambda_1 \dots \lambda_n$

Show
$$A = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}}_{=P} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{=D} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^T$$

and $A^T = A$

By diagonalization: $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^{-1}$

Since $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ is orthogonal, & square, $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^{-1} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^T$

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^T \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \overset{1}{v_1 \cdot v_1} & \overset{0}{v_1 \cdot v_2} & \cdots & v_1 \cdot v_n \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \cdots & v_2 \cdot v_n \\ \vdots & \vdots & & \vdots \\ v_n \cdot v_1 & v_n \cdot v_2 & \cdots & v_n \cdot v_n \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

Singular Value Decomposition

$(A = \underbrace{U}_{\text{orthogonal}}\underbrace{\Sigma}_{\text{diagonal}}\underbrace{V^T}_{\text{orthogonal}})$

Idea: A^TA is always symmetric

$\Rightarrow$ A^TA has an orthonormal basis of eigenvectors & its eigenvals are all real

Thm The eigenvals of A^TA are nonnegative real numbers

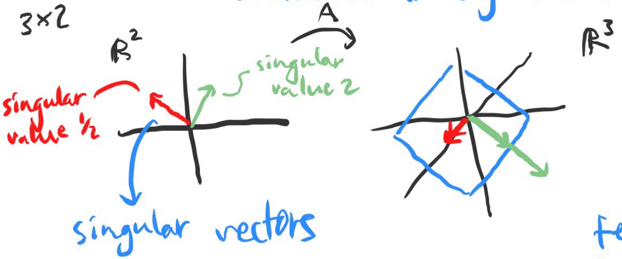
singular values of A

A^TA eigenvals $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2 \geq 0$

orthonormal eigenvectors $\vec{v}_1, \dots, \vec{v}_n$

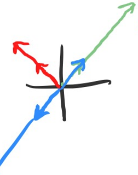
right singular vectors

A 3×2



Singular vector w/ largest singular value tells you direction that A expands the most

③ Suppose A is a 2×2 matrix and $\vec{v}_1, \vec{v}_2$ are eigenvectors for A with eigenvals $3, 2$ and they are orthonormal. Find the maximum value of $\|A\vec{v}\|$ over all unit vectors $\vec{v}$. $\boxed{3}$



Hint: try writing $\vec{v}$ as a linear combination of $\vec{v}_1$ & $\vec{v}_2$ & calculating $\|\vec{v}\|^2$ & $\|A\vec{v}\|^2$

Let $\vec{v} \in \mathbb{R}^2$. Because $\vec{v}_1, \vec{v}_2$ form a basis for $\mathbb{R}^2$, there are $a, b \in \mathbb{R}$ s.t. $\vec{v} = a\vec{v}_1 + b\vec{v}_2$

$$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} = (a\vec{v}_1 + b\vec{v}_2)\cdot(a\vec{v}_1 + b\vec{v}_2) = a^2\vec{v}_1\cdot\vec{v}_1 + ab\vec{v}_1\cdot\vec{v}_2 + ba\vec{v}_2\cdot\vec{v}_1 + b^2\vec{v}_2\cdot\vec{v}_2 = a^2 + b^2$$

$$\|A\vec{v}\|^2 = \|A(a\vec{v}_1 + b\vec{v}_2)\|^2 = \|a\cdot A\vec{v}_1 + bA\vec{v}_2\|^2 = \|3a\vec{v}_1 + 2b\vec{v}_2\|^2 = (3a)^2 + (2b)^2 = 9a^2 + 4b^2$$

largest when $a^2 = 1$ $b^2 = 0$ (assuming $a^2+b^2=1$)

If $\vec{v}_1 \dots \vec{v}_n$ is a basis of orthonormal eigenvectors of A with real eigenvalues then the direction in which A expands the most is the eigenvector w/ largest eigenvalue