Review Suppose $T : \text{span}\{\sin(x), \cos(x)\} \to \mathbb{R}^3$ defined by $T(f) = \begin{bmatrix} f(0) \\ f(\pi) \\ f(\pi/2) \end{bmatrix}$. Pick bases for the domain and codomain of $T$ and write the matrix for $T$ relative to those bases.

Basis for $\text{span}\{\sin(x), \cos(x)\}$: $B = \{\sin(x), \cos(x)\}$

Basis for $\mathbb{R}^3$: standard basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$T(\sin(x)) = \begin{bmatrix} \sin(0) \\ \sin(\pi) \\ \sin(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(\cos(x)) = \begin{bmatrix} \cos(0) \\ \cos(\pi) \\ \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$_{\text{std}}[T]_B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

# Announcements

① I am recording my 8-9 am discussion section
If you want access to the recording, email me

② Exam next Tuesday: Monday 8 pm PDT - Wed 8 am

Review session: Sunday

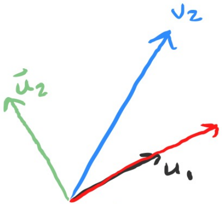
Plan for the next week

Friday Normal section
Sunday Review: half review problems, half questions
Monday Questions
Wednesday Go over exam solutions

Friday I'll give more details abt review session

# Gram-Schmidt

**Goal:** Given a basis for a subspace, find an orthogonal basis for the same subspace




**Algorithm:** Given $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

① $\vec{u}_1 = \vec{v}_1$

② $\vec{u}_2 = \vec{v}_2 - \operatorname{proj}_{\operatorname{span}\{\vec{u}_1\}}(\vec{v}_2)$

$$= \vec{v}_2 - \left( \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 \right)$$

③ $\vec{u}_3 = \vec{v}_3 - \operatorname{proj}_{\operatorname{span}\{\vec{u}_1, \vec{u}_2\}}(\vec{v}_3)$

$$= \vec{v}_3 - \left( \frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 \right)$$

$\vdots$

ⓝ $\vec{u}_n = \vec{v}_n - \operatorname{proj}_{\operatorname{span}\{\vec{u}_1 \ldots \vec{u}_{n-1}\}}(\vec{v}_n)$

At step $k+1$

① $\operatorname{span}\{\vec{u}_1 \ldots \vec{u}_k\} = \operatorname{span}\{\vec{v}_1 \ldots \vec{v}_k\}$

② $\vec{u}_1, \ldots, \vec{u}_k$ are orthogonal

$\vec{v}_{k+1} - \operatorname{proj}_{\operatorname{span}\{u_1, \ldots, u_k\}}(\vec{v}_{k+1})$

① Find an orthogonal basis for Col(A)

$$A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 5 & 4 \\ 3 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 10 \\ 4 \\ 8 \\ 3 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\text{span}\{\vec{u}_1\}}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \vec{v}_2 - \frac{30}{15}\vec{u}_1 = \begin{bmatrix} 3-2\cdot 1 \\ 5-2\cdot 2 \\ 5-2\cdot 3 \\ 2-2\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -\ldots \\ 0 \end{bmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{\text{span}\{\vec{u}_1, \vec{u}_2\}}(\vec{v}_3) = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 - \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2$$

$$= \vec{v}_3 - \frac{45}{15}\vec{u}_1 - \frac{6}{3}\vec{u}_2 = \begin{bmatrix} 10-3\cdot 1-2\cdot 1 \\ 4-3\cdot 2-2\cdot 1 \\ 8-3\cdot 3-2\cdot(-1) \\ 3-3\cdot 1-2\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

$\vec{v}_2 \cdot \vec{u}_1 = 3\cdot 1 + 5\cdot 2 + 5\cdot 3 + 2\cdot 1 = 3 + 10 + 15 + 2 = 30$

$\vec{u}_1 \cdot \vec{u}_1 = 1^2 + 2^2 + 3^2 + 1^2 = 1 + 4 + 9 + 1 = 15$

$\vec{v}_3 \cdot \vec{u}_1 = 10\cdot 1 + 4\cdot 2 + 8\cdot 3 + 3\cdot 1 = 10 + 8 + 24 + 3 = 45$

$\vec{v}_3 \cdot \vec{u}_2 = 10\cdot 1 + 4\cdot 1 + 8(-1) + 3\cdot 0 = 10 + 4 - 8 = 6$

$\vec{u}_2 \cdot \vec{u}_2 = 1^2 + 1^2 + (-1)^2 + 0^2 = 3$

## Least squares:

Goal: Given $A, \vec{b}$ s.t. $A\vec{x} = \vec{b}$ is inconsistent, find $\hat{x}$ such that $\|A\hat{x} - \vec{b}\|$ is as small as possible

### Algorithm 1:

① Use Gram-Schmidt to find an orthogonal basis for $Col(A)$

② Find $\hat{b} = \mathrm{proj}_{Col(A)}(\vec{b})$

③ Use row reduction to find $\hat{x}$ s.t. $A\hat{x} = \hat{b}$

**Algorithm 2:** Use row reduction to find $\hat{x}$ s.t. $A^T A\hat{x} = A^T\vec{b}$

Reason: $\vec{b} - A\hat{x}$ should be orthogonal to $Col(A)$

$$\Rightarrow A^T(\vec{b} - A\hat{x}) = \vec{0} \Rightarrow A^T A\hat{x} = A^T\vec{b}$$

① Suppose the Least squares sol'n to $A\vec{x} = \vec{b}$ is

$$\hat{x} = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}1 & 2 & 3\\-1 & 3 & 2\\1 & 1 & 2\end{bmatrix}$$

What is $\text{proj}_{Col(A)}(\vec{b})$?

$$A^T = \begin{bmatrix}1 & -1 & 1\\2 & 3 & 1\\3 & 2 & 2\end{bmatrix}$$

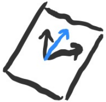
$\hat{x}$ s.t. $\|A\hat{x} - \vec{b}\|$ is as small as possible

$\Rightarrow A\hat{x} = \hat{b}$ in $Col(A)$ s.t. $\|\hat{b} - \vec{b}\|$ is as small as possible $= \text{proj}_{Col(A)}(\vec{b})$

$$A\hat{x} = \begin{bmatrix}1 & 2 & 3\\-1 & 3 & 2\\1 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}6\\4\\4\end{bmatrix}$$

---

If you run G-S on a linearly dependent set of vectors then G-S will give you an orthogonal basis for the span of the original set plus some $\vec{0}$s.



$v_1, v_2, v_3, v_4$

$v_3 \in \text{span}\{v_1, v_2\}$

$u_1 = v_1$

$u_2 = v_2 - \text{proj}_{\text{span}\{u_1\}}(v_2)$

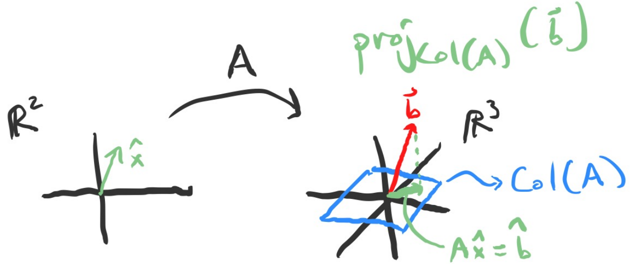
$u_3 = \vec{0}$

$u_4 = v_4 - \text{proj}_{\text{span}\{\vec{u}_1, u}$

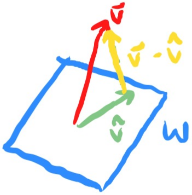
$\hat{x}$ is a vector s.t. $\|A\hat{x} - \vec{b}\|$ is as small as possible

could be any vector in $Col(A)$

$\Rightarrow$ $A\hat{x}$ is the vector in $Col(A)$ which is closest to $\vec{b}$



Property of $\text{proj}_W(\vec{v})$: it is the vector $\hat{v}$ in $W$ that makes $\|\hat{v} - \vec{v}\|$ as small as possible



Why is Null(A) orthogonal to Row(A)?

$$A\vec{x} = \begin{bmatrix} (1^{st} \text{ row of } A)\cdot\vec{x} \\ \vdots \\ (n^{th} \text{ row of } A)\cdot\vec{x} \end{bmatrix}$$

$\vec{x} \in \text{Null}(A)$ means $A\vec{x} = \vec{0}$ $\iff$ $\vec{x}$ is orthogonal to each row of $A$

$\iff$ $\vec{x}$ is orthogonal to all of Row(A)

# Transpose

$A^T$ = columns of $A$ are rows of $A^T$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Trick: $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = \vec{y}^T \vec{x} = \vec{y} \cdot \vec{x}$

Example: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix}$$

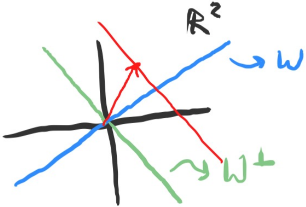
$1 \times 1$

# Orthogonal Complement

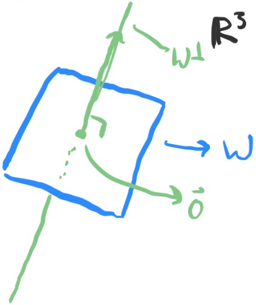
If $W$ is a subspace of $\mathbb{R}^n$ then

$$W^\perp = \{ \vec{u} \in \mathbb{R}^n \mid \text{for all } \vec{v} \in W,\ \vec{u} \cdot \vec{v} = 0 \}$$

Example




$$\dim(W) + \dim(W^\perp) = n$$



① $\{\vec{0}\}^\perp = \mathbb{R}^n$

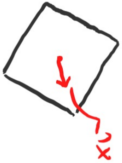
$\vec{0} \in \mathbb{R}^n$

② $(\mathbb{R}^n)^\perp = \{\vec{0}\}$. If $\vec{v}$ is nonzero then $\vec{v}\cdot\vec{v} \neq 0$

③ Suppose $W$ is a subspace of $\mathbb{R}^n$. What is $W \cap W^\perp$?

$\{\vec{0}\}$. If $\vec{v} \in W \cap W^\perp$ then $\vec{v}\cdot\vec{v} = 0 \Rightarrow \vec{v} = \vec{0}$

④ If $\vec{x} \in W$, what are $\text{proj}_W(\vec{x})$ and $\text{proj}_{W^\perp}(\vec{x})$?



$\text{proj}_W(\vec{x}) = \vec{x}$

$\text{proj}_{W^\perp}(\vec{x}) = \vec{0}$

$\vec{x} = \underbrace{\vec{0}}_{W^\perp} + \vec{x}$

orthogonal to everything in $W^\perp$

There's a unique way to write a vector $\vec{v}$ as $\underbrace{\text{something in } W}_{\text{projection onto } W}$ plus something orthogonal to $W$

$\vec{v} = \text{proj}_W(\vec{v}) + \text{proj}_{W^\perp}(\vec{v})$

The orthogonal complement of Col(A) is Null($A^T$)

$$\text{Col}(A)^\perp = \text{Null}(A^T)$$

$\vec{u} \in \text{Col}(A)^\perp \iff \vec{u}$ is orthogonal to all of the cols of $A$

$\iff \vec{u}$ is orthogonal to the rows of $A^T$

$\iff A^T \vec{u} = \vec{0}$

# Orthogonal Matrices

$\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$ is orthonormal if they are all orthogonal to each other and $\|\vec{v}_i\| = 1$ for each $i$.

Reminder: If $\vec{v}_1, \dots, \vec{v}_n$ orthogonal & nonzero then $\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \dots, \frac{\vec{v}_n}{\|\vec{v}_n\|}$ are orthonormal

An $n \times m$ matrix $U$ is called orthogonal if its columns are orthonormal

① Show that if $U$ is orthogonal then $U^TU = I_m$

$$U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix} \qquad U^T = \begin{bmatrix} — & u_1 & — \\ & \vdots & \\ — & u_m & — \end{bmatrix}$$

$$U^TU = U^T \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ U^T\vec{u}_1 & \cdots & U^T\vec{u}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} \overset{=1}{u_1 \cdot u_1} & \overset{=0}{u_1 \cdot u_2} & \\ \underset{=0}{u_2 \cdot u_1} & \overset{=1}{u_2 \cdot u_2} & \cdots \\ \underset{=0}{u_n \cdot u_1} & \overset{=0}{u_n \cdot u_2} & \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

---

If $U$ is square then $UU^T = I$

This is false if $U$ not square

$$\operatorname{proj}_W(\vec{x}) = \frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \dots \frac{x\cdot\vec{u}_n}{\vec{u}_n\cdot\vec{u}_n}\vec{u}_n$$

where $\{\vec{u}_1, \dots, \vec{u}_n\}$ are an orth. basis for $W$

Gives the same answer for any orthogonal basis for $W$

If col's of $U$ are orthogonal then

$U^T U =$ diagonal

If $U$ is orthogonal and square then

$U^{-1} = U^T \Rightarrow UU^T = I_n$ ← $U$ is called a unitary matrix

This doesn't work if $U$ is not square

Example:

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$U^T U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$UU^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

If $A$ is square and

$AB = I$ then

$BA = I$

③ If $U$ is orthogonal, show

$$\|U\vec{x}\| = \|\vec{x}\|$$

Hint: $\|\vec{x}\|^2 = \vec{x}^T\vec{x}$

$(AB)^T = B^T A^T$

Enough to show $\|U\vec{x}\|^2 = \|\vec{x}\|^2$

$$\begin{aligned}
\|U\vec{x}\|^2 &= (U\vec{x})^T(U\vec{x}) \\
&= \vec{x}^T \underbrace{U^T U}_{= I} \vec{x} \\
&= \vec{x}^T\vec{x} \\
&= \|\vec{x}\|^2
\end{aligned}$$