Dot Product

$\vec{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $\vec{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ vectors in $\mathbb{R}^n$

① Dot product: $\vec{u} \cdot \vec{v} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

② Norm/length/magnitude: $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{a_1^2 + \dots + a_n^2}$

③ Distance: $\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(a_1 - b_1)^2 + \dots + (a_n - b_n)^2}$

④ Orthogonal: $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

Useful facts:

① $\vec{u} \cdot (a\vec{v}_1 + b\vec{v}_2) = a(\vec{u} \cdot \vec{v}_1) + b(\vec{u} \cdot \vec{v}_2)$

② $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta \leftarrow$ angle between $\vec{u}$ and $\vec{v}$

③ Nonzero orthogonal vectors are always linearly independent

① $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$

a) $\|\vec{u}\| = \sqrt{1^2+2^2+(-3)^2} = \sqrt{1+4+9} = \sqrt{14}$

$\|\vec{v}\| = \sqrt{0^2+(-6)^2+4^2} = \sqrt{36+16} = \sqrt{52}$

b) $\text{dist}(\vec{u},\vec{v}) = \sqrt{(1-0)^2+[2-(-6)]^2+(-3-4)^2} = \sqrt{1^2+8^2+(-7)^2} = \sqrt{1+64+49} = \sqrt{114}$

c) $\vec{u}\cdot\vec{v} = 1\cdot 0 + 2\cdot(-6) + (-3)\cdot 4 = -12-12 = -24$

d) $\vec{u}\cdot(\vec{u}+2\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} = 14+2(-24) = -34$

e) Cosine of angle between $\vec{u}$ and $\vec{v}$ $= \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \dfrac{-24}{\sqrt{14\cdot 52}}$

f) Nonzero vector orthogonal to $\vec{u}$

Lots of answers possible. Just need $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $a\cdot 1+b\cdot 2+c\cdot(-3)=0$

E.g. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, etc.

g) Unit vector in the same direction as $\vec{u}$

$\dfrac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}$ works.

② $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$. Find a vector orthogonal to both $\vec{u}$ and $\vec{v}$.

Want $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\begin{cases} a \cdot 1 + b \cdot 2 + c \cdot (-3) = 0 \\ a \cdot 0 + b \cdot (-6) + c \cdot 4 = 0 \end{cases}$. Find a, b, c by row reduction!

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -6 & 4 \end{bmatrix} \xrightarrow{R_2 = -\frac{1}{6}R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2/3 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & -2/3 \end{bmatrix}$$

$x_1 = (5/3)x_3$

$x_2 = (2/3)x_3$

x_3 free

So one possible answer is $\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$

<u>Check</u>: $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot 5 + 2 \cdot 2 + (-3) \cdot 3 = 5 + 4 - 9 = 0$ ✓

$\begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} = 0 \cdot 5 + (-6) \cdot 2 + 4 \cdot 3 = -12 + 12 = 0$ ✓

③ True or False: any three nonzero orthogonal vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$ True

$\vec{u}_1, \vec{u}_2, \vec{u}_3 \in \mathbb{R}^3$ nonzero and orthogonal

$$\Downarrow$$

$\vec{u}_1, \vec{u}_2, \vec{u}_3$ linearly independent.

Any three linearly independent vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$ (by the invertible matrix thm, basically)

Side note: Why are $\vec{u}_1, \vec{u}_2, \vec{u}_3$ lin independent?

Suppose $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = \vec{0}$. We want to show $a=b=c=0$.

We have $0 = \vec{0}\cdot\vec{u} = (a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3)\cdot\vec{u}_1 = a\vec{u}_1\cdot\vec{u}_1 + b\cancelto{0}{\vec{u}_2\cdot\vec{u}_1} + c\cancelto{0}{\vec{u}_3\cdot\vec{u}_1}$

$= a\underbrace{\|\vec{u}_1\|^2}_{\text{nonzero since } \vec{u}_1 \neq \vec{0}}$

Hence $a=0$. b and c can be handled similarly.

④ True or false: If $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$ then it is orthogonal to $2\vec{v}+3\vec{w}$. True

$$\vec{u}\cdot(2\vec{v}+3\vec{w}) = 2\,\vec{u}\cdot\vec{v}+3\,\vec{u}\cdot\vec{w}$$

$$= 2\cdot 0 + 3\cdot 0$$

because $\vec{u}$ is orthogonal to $\vec{v}$ and $\vec{w}$.

$$= 0$$

More generally, $\vec{u}$ is orthogonal to everything in $\operatorname{span}\{\vec{v},\vec{w}\}$.