Similar Matrices Suppose A and B are $n \times n$ matrices. A and B are similar if:

Abstract definition: There is a vector space V, a linear transformation $T: V \to V$ and bases B_1, B_2 for V such that $A = {}_{B_1}[T]_{B_1}$ and $B = {}_{B_2}[T]_{B_2}$

Concrete definition: There is an invertible matrix P such that $A = PBP^{-1}$

think of P as $B_1 \xleftarrow{P} B_2$

① Suppose A and B are similar matrices and $\det(A) = 5$. What can you say about $\det(B)$?

There is some invertible P s.t. $A = PBP^{-1}$

$$5 = \det(A) = \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1})$$
$$= \det(P)\det(B)\frac{1}{\det(P)}$$
$$= \det(B) \Rightarrow \det(B) = 5$$

② Suppose A is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. What can you say about A?

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Why? There is some invertible P s.t. $A = P\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$

$$A = P\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = P\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The all-zeros matrix times any other matrix is just the all-zeros matrix.

③ Suppose A is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What can you say about A?

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Why? The identity matrix times any matrix just gives that matrix. So we have

$$A = P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}P^{-1} = PP^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Diagonalization An $n \times n$ matrix A is diagonalizable if it is similar to a diagonal matrix.

$=$ There is a basis for $\mathbb{R}^n$ consisting of eigenvectors for A

Algorithm to diagonalize a matrix A:

1. Find the eigenvalues of A as roots of the characteristic polynomial, $\chi_A(t)$.
2. For each eigenvalue, λ, find a basis for the eigenspace associated to λ (i.e. a basis for $Null(A - \lambda I_n)$)
3. If the dimensions of the eigenspaces sum to n then A is diagonalizable. Otherwise it's not.

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{\text{eigenvectors of } A} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}}_{\text{corresponding eigenvalues of } A} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^{-1}$$

① Try to diagonalize:

a) $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$

① $\chi_A(t) = \det(A - tI_2) = \det\left(\begin{bmatrix} 2-t & 1 \\ 0 & 3-t \end{bmatrix}\right)$

$= (2-t)(3-t)$

Eigenvalues: 2, 3

② $\underline{\text{Basis for } E_2}$: $\text{Null}(A - 2I_2) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$A - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \left.\begin{matrix} x_1 \text{ free} \\ x_2 = 0 \end{matrix}\right\} x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{\text{Basis for } E_3}$: $\text{Null}(A - 3I_2) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$A - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 = -R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad \left.\begin{matrix} x_1 = x_2 \\ x_2 \text{ free} \end{matrix}\right\} x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

③ $\dim(E_2) + \dim(E_3) = 2$ So A $\underline{\text{is}}$ diagonalizable

b) $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

① Eigenvalues of B: 2

$$\chi_B(t) = \det \begin{bmatrix} 2-t & 1 \\ 0 & 2-t \end{bmatrix} = (2-t)^2$$

② Basis for E_2: $\mathrm{Null}(B - 2I_2) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$$B - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \left.\begin{matrix} x_1 \text{ free} \\ x_2 = 0 \end{matrix}\right\} \quad x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

③ $\dim(E_2) = 1$ So B is not diagonalizable.

② Find a 2×2 matrix A such that:

- $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 5
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue -1

Let $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\vec{x}) = A\vec{x}$. Note that B is a basis for $\mathbb{R}^2$ and the matrix for T relative to the basis B is $\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$. Since $[T]_{std} = A$, this tells us that

$$A = \underset{std \leftarrow B}{P} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \underset{B \leftarrow std}{P} = \underset{std \leftarrow B}{P} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \underset{std \leftarrow B}{P}^{-1}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 8 & -9 \\ 3 & -4 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 8 & -9 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 24-9 \\ 9-4 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -9 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8-9 \\ 3-4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Why Diagonalize?

① $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

a) $A^{100} = \begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix} = \begin{bmatrix} 3^{100} & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\vdots$$

$$A^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix}$$

b) $B^{100} = \frac{1}{2}\begin{bmatrix} 3^{100}+1 & 3^{100}-1 \\ 3^{100}-1 & 3^{100}+1 \end{bmatrix}$

$$B^{100} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} A \overbrace{\cancel{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}}^{I_2} A \overbrace{\cancel{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \cdots}}^{I_2} \cdots \overbrace{\cancel{\cdots \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}}^{I_2} A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} A^{100} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^{100} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{100} & 1 \\ 3^{100} & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(3^{100}+1) & \frac{1}{2}(3^{100}-1) \\ \frac{1}{2}(3^{100}-1) & \frac{1}{2}(3^{100}+1) \end{bmatrix}$$

② $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^{2021} = \begin{bmatrix} 2^{2021} & 3^{2021} - 2^{2021} \\ 0 & 3^{2021} \end{bmatrix}$

In a previous problem, we found:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

So $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^{2021} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{2021} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{2021} & 0 \\ 0 & 3^{2021} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{2021} & 3^{2021} \\ 0 & 3^{2021} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{2021} & 3^{2021} - 2^{2021} \\ 0 & 3^{2021} \end{bmatrix}$$

<u>Challenge</u>: Find $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{2021}$

Extra Problems

① What is the maximum number of eigenvalues a 5×5 matrix can have? Answer: 5

Each eigenvalue has an eigenvector and eigenvectors with different eigenvalues are always linearly independent. Since there can be at most 5 linearly independent vectors in $\mathbb{R}^5$, a 5×5 matrix can have at most 5 eigenvalues. Also, there are 5×5 matrices which have this many eigenvalues.

E.g.
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

What is the minimum number a 5×5 matrix can have and still be diagonalizable? Answer: 1

It has to have at least one to be diagonalizable.
But it can have exactly 1. E.g.

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \leftarrow$$

diagonalizable because it is a diagonal matrix.

② True or False:

a) Every 5×5 matrix with 5 distinct eigenvalues is diagonalizable. True

Each eigenspace has dimension at least 1 so their dimensions sum to 5, hence the matrix is diagonalizable.

b) Every invertible matrix is diagonalizable False

Counterexample: $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

c) Every diagonalizable matrix is invertible False

Counterexample: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ($\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ also works!)

(Note: A is invertible $\Leftrightarrow$ 0 is not an eigenvalue of A)

d) If $A \neq 0$ and $A^2 = 0$ then A is not diagonalizable

True

If 0 is the only eigenvalue of A & A is diagonalizable then $A = 0$. So to be diagonalizable, A must have a nonzero eigenvalue, λ. But if $A\vec{v} = \lambda\vec{v}$ and $\vec{v} \neq 0$ then

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda^2\vec{v} \neq \vec{0}$$

e) Every 2×2 matrix with more than one eigenvalue is diagonalizable True

See part (a)

f) Every upper triangular matrix is diagonalizable False

Counterexample: $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$