The Matrix of a Linear Transformation

① $T: M_{2\times 2} \to M_{2\times 2}$ defined by $T(B) = AB$

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$$

a) Find a basis for $M_{2\times 2}$

Recall that a basis for a vector space is a set of vectors which span the whole space and are linearly independent.

One basis for $M_{2\times 2}$ is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

To see this set spans all of $M_{2\times 2}$, note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

b) Write the matrix for T relative to the basis from part (a).

To find this matrix, we need to evaluate T on each vector in $\mathcal{B}$ and write the result as a coordinate vector relative to $\mathcal{B}$.

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 10 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 5 \\ 0 \\ 10 \end{bmatrix}$$

$${}_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \\ 2 & 0 & 10 & 0 \\ 0 & 2 & 0 & 10 \end{bmatrix}$$

c) Is T one-to-one? Onto?

T 1-to-1 $\Leftrightarrow$ ${}_B[T]_B$ has a pivot in every column in REF

T onto $\Leftrightarrow$ ${}_B[T]_B$ has a pivot in every row in REF

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \\ 2 & 0 & 10 & 0 \\ 0 & 2 & 0 & 10 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 10 \end{bmatrix} \xrightarrow{R_4 = R_4 - 2R_2}$$

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ REF}$$

No pivots in columns 3 & 4 $\Rightarrow$ T is not 1-to-1

No pivots in rows 3 & 4 $\Rightarrow$ T is not onto

d) Find bases for $\ker(T)$ and $\text{range}(T)$.

$\ker(T) \sim \text{Null}({}_B[T]_B)$

$\text{range}(T) \sim \text{Col}({}_B[T]_B)$

$\text{Null}({}_B[T]_B)$: $\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -5x_3$
$x_2 = -5x_4$
x_3 free
x_4 free

Solutions to homogeneous equation:

$$\begin{bmatrix} -5x_3 \\ -5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathrm{Null}({}_B[T]_B) = \mathrm{span}\left\{ \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To find a basis for $\ker(T)$, translate these vectors back to $M_{2\times 2}$.

$$\ker(T) = \mathrm{span}\left\{ \begin{bmatrix} -5 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -5 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathrm{Col}({}_B[T]_B): \quad \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \\ 2 & 0 & 10 & 0 \\ 0 & 2 & 0 & 10 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

$$\Rightarrow \mathrm{Col}({}_B[T]_B) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

To find a basis for $\mathrm{range}(T)$, translate these vectors back to $M_{2\times 2}$

$$\mathrm{range}(T) = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right\}$$

Change of Basis

① Suppose V is a 2-dimensional vector space and $B = \{\vec{v}_1, \vec{v}_2\}$, $C = \{\vec{u}_1, \vec{u}_2\}$ are bases for V such that

$$\vec{u}_1 = \vec{v}_1 + \vec{v}_2$$
$$\vec{u}_2 = \vec{v}_1 - \vec{v}_2$$

a) If $\vec{w} \in V$ such that $[\vec{w}]_C = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, what is $[\vec{w}]_B$?

$[\vec{w}]_C = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow$

$$\begin{aligned} \vec{w} &= 2\cdot\vec{u}_1 + 3\cdot\vec{u}_2 \\ &= 2(\vec{v}_1 + \vec{v}_2) + 3(\vec{v}_1 - \vec{v}_2) \\ &= 5\vec{v}_1 - \vec{v}_2 \end{aligned}$$

So $[\vec{w}]_B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

b) Find a matrix A such that for all $\vec{x} \in V$, $A[\vec{x}]_C = [\vec{x}]_B$.

Suppose $[\vec{x}]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then

$$\begin{aligned} \vec{x} &= x_1\cdot\vec{u}_1 + x_2\cdot\vec{u}_2 \\ &= x_1(\vec{v}_1 + \vec{v}_2) + x_2(\vec{v}_1 - \vec{v}_2) \\ &= (x_1 + x_2)\vec{v}_1 + (x_1 - x_2)\vec{v}_2 \end{aligned}$$

Hence $[\vec{x}]_B = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$

This tells us that $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$

1^{st} col. of A $= A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2^{nd} col of A $= A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

The matrix A is the change-of-coordinates matrix from C to B (aka "change of basis matrix"), written $\underset{B \leftarrow C}{P}$

c) Find a matrix D s.t. for all $\vec{x} \in V$, $D[\vec{x}]_B = [\vec{x}]_C$.

In other words, find $\underset{C \leftarrow B}{P}$.

Two approaches.

Approach 1: Guess-and-check

We can repeat the method of the solution to part (b) if we can write $\vec{v}_1$ and $\vec{v}_2$ as linear combinations of $\vec{u}_1$ and $\vec{u}_2$.

Notice

$$\vec{u}_1 + \vec{u}_2 = (\vec{v}_1 + \vec{v}_2) + (\vec{v}_1 - \vec{v}_2) = 2\vec{v}_1$$
$$\vec{u}_1 - \vec{u}_2 = (\vec{v}_1 + \vec{v}_2) - (\vec{v}_1 - \vec{v}_2) = 2\vec{v}_2$$

Hence

$$\vec{v}_1 = \tfrac{1}{2}\vec{u}_1 + \tfrac{1}{2}\vec{u}_2$$
$$\vec{v}_2 = \tfrac{1}{2}\vec{u}_1 - \tfrac{1}{2}\vec{u}_2$$

So

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Approach 2: Systematic approach

Notice that $\underset{C \leftarrow B}{P} = \underset{B \leftarrow C}{P}^{-1}$

So we just need to invert $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{2}R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right]$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

② $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ $C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Find $\underset{C \leftarrow B}{P}$.

Two approaches

Approach 1: $\underset{C \leftarrow B}{P} = \underset{C \leftarrow std}{P} \cdot \underset{std \leftarrow B}{P} = \underset{std \leftarrow C}{P}^{-1} \cdot \underset{std \leftarrow B}{P}$

$$\underset{std \leftarrow B}{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underset{std \leftarrow C}{P} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{3}R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right]$$

Check: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} = \begin{bmatrix} -1/3 + 4/3 & 2/3 - 2/3 \\ -2/3 + 2/3 & 4/3 - 1/3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1 \\ 1/3 & 1 \end{bmatrix}$$

Approach 2: The formula for $\underset{C \leftarrow B}{P}$ when $B = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $C = \{\vec{c}_1, \dots, \vec{c}_n\}$ is

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} | & | & & | \\ [\vec{b}_1]_C & [\vec{b}_2]_C & \cdots & [\vec{b}_n]_C \\ | & | & & | \end{bmatrix}$$

So to find $\underset{C \leftarrow B}{P}$ we need to find coordinate vectors in the basis C for $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

I.e. we need to write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as linear combinations of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

I.e. we need to find $a, b, c, d \in \mathbb{R}$ s.t.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \end{bmatrix} + d \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & -1 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & -3 & -1 & -3 \end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/3 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/3 & -1 \\ 0 & 1 & 1/3 & 1 \end{array}\right]$$

So

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 1/3 & -1 \\ 1/3 & 1 \end{bmatrix}$$

In general if $B = \{\vec{b}_1, \dots, \vec{b}_n\}$, $C = \{\vec{c}_1, \dots, \vec{c}_n\}$ are bases for $\mathbb{R}^n$, you can find $\underset{C \leftarrow B}{P}$ by row reducing

$$\left[\begin{array}{ccc|ccc} | & & | & | & & | \\ \vec{c}_1 & \cdots & \vec{c}_n & \vec{b}_1 & \cdots & \vec{b}_n \\ | & & | & | & & | \end{array}\right] \rightsquigarrow \left[\begin{array}{c|c} I_n & \underset{C \leftarrow B}{P} \end{array}\right]$$