Review

1. True or False: The set of invertible $3 \times 3$ matrices is a subspace of $M_{3 \times 3}$ (i.e. of the vector space of all $3 \times 3$ matrices).

More Practice with Coordinates

1. Consider the following three polynomials in $\mathbb{P}_2$.
   \[ p(x) = -2x^2 + 4x + 4 \]
   \[ q(x) = 3x^2 + 6x - 2 \]
   \[ r(x) = -2x^2 + x + 3 \]
   (a) What is the dimension of $\text{span}\{p(x), q(x), r(x)\}$?
   (b) Find a basis for $\text{span}\{p(x), q(x), r(x)\}$.

The Matrix of a Linear Transformation Relative to a Basis

1. Let $T: \mathbb{P}_2 \rightarrow \mathbb{R}^2$ be the linear transformation defined by
   \[ T(p) = \begin{bmatrix} \int_0^2 p(x) \, dx \\ \int_1^3 p(x) \, dx \end{bmatrix} \]
   You do not need to check that $T$ is a linear transformation.
   (a) Let $\mathcal{B}$ be the basis $1, x, x^2$ for $\mathbb{P}_2$ and let $\mathcal{C}$ be the basis $[1, 0], [0, 1]$ for $\mathbb{R}^2$ (i.e. the standard basis). Find $\mathcal{C}[T]_{\mathcal{B}}$.
   (b) Find a basis for the range of $T$.
   (c) What is the dimension of the kernel of $T$?
   (d) Find a nontrivial element of the kernel of $T$. Try graphing the polynomial that you found.

2. You may have heard before that knowing the value of a quadratic polynomial on three points completely determines the polynomial. Let’s use linear algebra to see how to do this.
   (a) Let $T: \mathbb{P}_2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by
      \[ T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix} \]
      Calculate $T(1), T(x), \text{and } T(x^2)$.
   (b) Let $\mathcal{B}$ be the basis $1, x, x^2$ for $\mathbb{P}_2$ and let $\mathcal{C}$ be the standard basis for $\mathbb{R}^3$. Find $\mathcal{C}[T]_{\mathcal{B}}$.
   (c) Check that $T$ is invertible and find a matrix representing the inverse of $T$.
   (d) Use your answer to part (c) to find a polynomial $p$ such that $p(0) = 10, p(1) = 5$, and $p(2) = -3$.
   (e) **Challenge problem:** Find a formula for the unique quadratic polynomial $p$ such that $p(a_0) = b_0, p(a_1) = b_1$, and $p(a_2) = b_2$ (assuming that $a_0, a_1, a_2$ are all distinct).