① Let's use linear algebra to find a solution to the differential equation $y'' + 2y' - 5y = 3\sin(x) - 2\cos(x)$

a) Let $V = \text{span}\{\sin(x), \cos(x)\}$. Show that $\sin(x), \cos(x)$ is a basis for V

Remember that a basis for a vector space is a list of vectors in the vector space such that:

① The span of the vectors is the entire vector space

② The vectors are linearly independent

Let's check both of these here.

① $\text{span}\{\sin(x), \cos(x)\} = V$ True by definition of V

② $\sin(x), \cos(x)$ lin. ind.

Suppose $a \cdot \sin(x) + b \cdot \cos(x) = 0$. We need to show $a = b = 0$

$$a\sin(x) + b\cos(x) = 0 \Rightarrow \begin{cases} a\sin(0) + b\cos(0) = 0 \Rightarrow b = 0 \\ a\sin(\pi/2) + b\cos(\pi/2) = 0 \Rightarrow a = 0 \end{cases}$$

b) Write the coordinate vector of $3\sin(x) - 2\cos(x)$ in the basis $\mathcal{B} = \{\sin(x), \cos(x)\}$

Remember that if $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis for a vector space V then the coordinate vector of a vector $\vec{v} \in V$ is the vector $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$ such that $\vec{v} = a_1 \cdot \vec{b}_1 + \dots + a_n \cdot \vec{b}_n$.

So how do we find a coordinate vector for $\vec{v} \in V$?

1. Write $\vec{v}$ as a linear combination of the basis vectors
2. Take the weights of this linear combination and make them the entries of a vector in $\mathbb{R}^n$

In this case, $3\sin(x) - 2\cos(x)$ is already written as a linear combination of $\sin(x)$ and $\cos(x)$, so the coordinate vector is $[3\sin(x) - 2\cos(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

c) Let $T: V \to V$ be the linear transformation defined by

$$T(f) = \frac{d^2 f}{dx^2} + 2\frac{df}{dx} - 5f$$

Find the matrix for T in the basis $\beta = \{\sin(x), \cos(x)\}$

To find the matrix for T, we evaluate T on each basis vector and write the result as a coordinate vector in this basis.

$$T(\sin(x)) = -\sin(x) + 2\cos(x) - 5\sin(x) = -6\sin(x) + 2\cos(x)$$

$$T(\cos(x)) = -\cos(x) - 2\sin(x) - 5\cos(x) = -2\sin(x) - 6\cos(x)$$

$$[T(\sin(x))]_\beta = \begin{bmatrix} -6 \\ 2 \end{bmatrix} \qquad \text{So } {}_\beta[T]_\beta = \begin{bmatrix} -6 & -2 \\ 2 & -6 \end{bmatrix}$$

$$[T(\cos(x))]_\beta = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

d) Let A be the matrix from part (c) and $\vec{v}$ the vector in $\mathbb{R}^2$ from part (b). Find a solution to $A\vec{x} = \vec{v}$.

$$\begin{bmatrix} -6 & -2 \\ 2 & -6 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -6 & -2 & 3 \\ 2 & -6 & -2 \end{array}\right] \xrightarrow{R_2 = 3R_2 + R_1} \left[\begin{array}{cc|c} -6 & -2 & 3 \\ 0 & -20 & -3 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{20} R_2}$$

$$\left[\begin{array}{cc|c} -6 & -2 & 3 \\ 0 & 1 & 3/20 \end{array}\right] \xrightarrow{R_1 = R_1 + 2R_2} \left[\begin{array}{cc|c} -6 & 0 & 66/20 \\ 0 & 1 & 3/20 \end{array}\right] \xrightarrow{R_1 = -\frac{1}{6} R_1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -11/20 \\ 0 & 1 & 3/20 \end{array}\right] \qquad \begin{array}{l} x_1 = -11/20 \\ x_2 = 3/20 \end{array} \qquad \begin{bmatrix} -11/20 \\ 3/20 \end{bmatrix}$$

e) Use the answer to part (d) to find a solution to the differential equation $y'' + 2y' - 5y = 3\sin(x) - 2\cos(x)$

A solution to $y'' + 2y' - 5y = 3\sin(x) - 2\cos(x)$ is a function $f(x)$ such that $T(f) = 3\sin(x) - 2\cos(x)$.

$T(f) = 3\sin(x) - 2\cos(x)$ has a solution in V if and only if $A\vec{x} = \vec{v}$ has a solution

We know from part (d) that $A\vec{x} = \vec{v}$ does have a solution: $\begin{bmatrix} -11/20 \\ 3/20 \end{bmatrix}$. This is the coordinate vector of a solution to $T(f) = 3\sin(x) - 2\cos(x)$.

The solution is:

$$(-11/20)\sin(x) + (3/20)\cos(x)$$

(2) Is $\{\sin^2(x), \cos^2(x), 1\}$ a basis for $\text{span}\{\sin^2(x), \cos^2(x), 1\}$?

2 things to check:

(1) Do $\sin^2(x), \cos^2(x), 1$ span all of $\text{span}\{\sin^2(x), \cos^2(x), 1\}$? Yes.

(2) Are $\sin^2(x), \cos^2(x), 1$ linearly independent?

No. $\sin^2(x) + \cos^2(x) = 1$ so

$$\sin^2(x) + \cos^2(x) - 1 = 0.$$

Thus there is a linear combination of $\sin^2(x), \cos^2(x), 1$ which is equal to 0 even though not all coefficients are 0.

③ Write the coordinate vector of $p(x) = x^2 - 1$ in the basis $\mathcal{B} = \{1, x, x^2 + x + 2\}$ for $\mathbb{P}_2$.

Method 1: Guess and check

$$x^2 - 1 = \underline{1} \cdot (x^2 + x + 2) \underline{-1} \cdot x \underline{-3} \cdot 1$$

$$\Longrightarrow \boxed{[p(x)]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}}$$

Method 2: Translate to $\mathbb{R}^3$ using a nicer basis & solve with row reduction

Basis $C = \{x^2, x, 1\}$

$$[1]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad [x]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad [x^2 + x + 2]_C = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad [p(x)]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Write $p(x)$ as a linear combination of $1, x, x^2 + x + 2$ $\Longrightarrow$ Write $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 \end{array}\right] \xrightarrow{\text{switch } R_1 \text{ \& } R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(4) If the coordinate vector of a polynomial $q(x) \in \mathbb{P}_2$ in the basis $\mathcal{B} = \{1, x, x^2+x+2\}$ is $[q(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, what is $q(x)$?

Remember that the coordinate vector of $q(x)$ consists of the weights of a linear combination of the basis vectors that is equal to $q(x)$.

$$[q(x)]_{\mathcal{B}} = \begin{bmatrix} \underline{1} \\ \underline{3} \\ \underline{-1} \end{bmatrix} \Rightarrow q(x) = \underline{1} \cdot 1 + \underline{3} \cdot x + \underline{(-1)} \cdot (x^2 + x + 2)$$

$$= -x^2 + 2x + 3$$