Abstract vector spaces

Any set where you can add elements to each other, multiply them by scalars & where the operations obey certain rules

1. There is a 0 vector — i.e. an element of the set such that for all x, $x + 0 = x$
2. $x + y = y + x$ for all x, y
3. $r \cdot (x + y) = r \cdot x + r \cdot y$
4.

⋮

① Which of these are vector spaces?

a) The set of sequences of real numbers that converge to 0 Yes. $(0,0,0,\ldots)$

$(a_0, a_1, a_2, \ldots) + (b_0, b_1, b_2, \ldots) = (a_0+b_0, a_1+b_1, a_2+b_2, \ldots)$

$c \cdot (a_0, a_1, a_2, \ldots) = (c \cdot a_0, c \cdot a_1, \ldots)$ $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

b) The set of sequences of real numbers that converge to 1 No.

$(\frac{1}{2}, 3/4, 7/8, \ldots) + (1,1,1,\ldots) = (3/2, 7/4, 15/8, \ldots)$

→ converges to 2

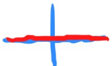
c) $\{a, b, c, d, \ldots, x, y, z\}$ No.

No obvious definition of addition or scalar multiplication.

d) The set of 2×3 matrices which are in RREF No.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ → not in RREF

e) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\}$ Yes. $(f+g)' = f' + g'$

 → constantly 0 function

f, g differentiable then so are $f+g$, $c \cdot f$

Examples of vector spaces

① Set of sequences of real numbers

set of sequences which converge to 0 is a linear subspace of this vector space

set of sequences which converge to 1 is <u>not</u> a linear subspace

② Set of functions from $\mathbb{R}$ to $\mathbb{R}$

set of differentiable functions $\mathbb{R} \to \mathbb{R}$ is a linear subspace

③ set of polynomials with real coefficients

set of polynomials with degree at most n ($\mathbb{P}_n$)

④ set of $n \times m$ matrices ($M_{n \times m}$)

set of $n \times m$ matrices in RREF is <u>not</u> a linear subspace

② Are $1,\ x^2,\ 3x^2-2$ linearly independent? No.

(elements of $\mathbb{P}_2$)

$$-2\cdot(1)+3\cdot(x^2)-1\cdot(3x^2-2)=0$$

③ Is $(1,0,1,0,\ldots)$ in $\operatorname{span}\{(1,1,1,\ldots),(1,-1,1,-1,\ldots)\}$

Yes.

$$(1,0,1,0,\ldots)=\frac{1}{2}(1,1,1,\ldots)+\frac{1}{2}(1,-1,1,-1,\ldots)$$

④ Are these matrices lin. ind.? No.

$$\begin{bmatrix}1&0\\0&-1\end{bmatrix},\ \begin{bmatrix}0&0\\1&0\end{bmatrix},\ \begin{bmatrix}0&1\\0&0\end{bmatrix},\ \begin{bmatrix}-1&-1\\1&1\end{bmatrix}$$

(elements of $M_{2\times2}$) $\cong \mathbb{R}^4$

$$-1\cdot\begin{bmatrix}1&0\\0&-1\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}+(-1)\begin{bmatrix}0&1\\0&0\end{bmatrix}-1\cdot\begin{bmatrix}-1&-1\\1&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$