① $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear transformation

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = ?$$

$$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}6\\3\end{bmatrix} \qquad \tfrac{1}{3}\begin{bmatrix}1\\2\end{bmatrix} + \tfrac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} = \tfrac{1}{3}\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\tfrac{1}{3}\begin{bmatrix}1\\2\end{bmatrix} + \tfrac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}\right) = \tfrac{1}{3}T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + \tfrac{1}{3}T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$$

$$= \tfrac{1}{3}\begin{bmatrix}2\\4\end{bmatrix} + \tfrac{1}{3}\begin{bmatrix}6\\3\end{bmatrix} = \begin{bmatrix}8/3\\7/3\end{bmatrix}$$

② $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = ?$

Want to write $\begin{bmatrix}x\\y\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & 2 & x\\ 2 & 1 & y\end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|c}1 & 2 & x\\ 0 & -3 & y-2x\end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{1}{3}R_2} \left[\begin{array}{cc|c}1 & 2 & x\\ 0 & 1 & \frac{-y+2x}{3}\end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cc|c}1 & 0 & x + \frac{2y-4x}{3}\\ 0 & 1 & \frac{2x-y}{3}\end{array}\right] \quad = \frac{2y-x}{3}$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \frac{2y-x}{3}\begin{bmatrix}2\\4\end{bmatrix} + \frac{2x-y}{3}\begin{bmatrix}6\\3\end{bmatrix} = \begin{bmatrix}(4y-2x+12x-6y)/3\\(8y-4x+6x-3y)/3\end{bmatrix}$$

③ Find the standard matrix for T.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} (10x-2y)/3 \\ (2x+5y)/3 \end{bmatrix} = \begin{bmatrix} 10/3\, x - 2/3\, y \\ 2/3\, x + 5/3\, y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 10/3 \\ 2/3 \end{bmatrix} \qquad [T]_{std} = \begin{bmatrix} 10/3 & -2/3 \\ 2/3 & 5/3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2/3 \\ 5/3 \end{bmatrix} \qquad A = [T]_{std}$$

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \qquad A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Rightarrow A\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 3 \end{bmatrix}$$

invertible

$$A = \begin{bmatrix} 2 & 6 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$

tells you how to write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

④ Suppose $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$ and $\vec{u}_1, \vec{u}_2$ are any vectors in $\mathbb{R}^n$. Show that there is a lin. transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^n \quad \text{s.t.} \quad T(\vec{v}_1) = \vec{u}_1$$
$$T(\vec{v}_2) = \vec{u}_2$$

Define $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$

tells us where to send $\vec{v}_1, \vec{v}_2$

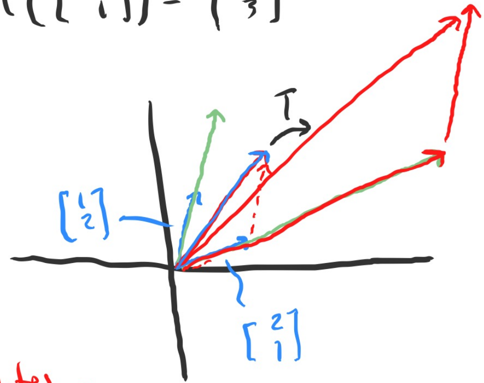
tells us how to write $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2$

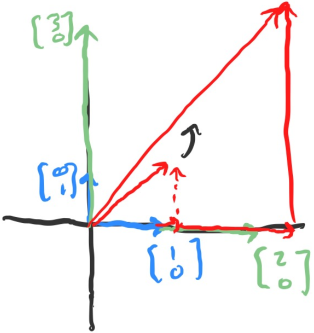
$A = [T]_{std}$ $\quad A \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$

invertible b/c $\{\vec{v}_1, \vec{v}_2\}$ is a basis

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix}$$

$$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}6\\3\end{bmatrix}$$





translates from $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$ basis to std basis

$$\begin{bmatrix}2 & 6\\4 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}^{-1}$$

$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}^{-1}$$

translates from std basis to $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$ basis

std matrix $\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$