Math 54 Midterm 2 Review

1. Which of the following are vector spaces? (For an extra challenge, for each set that is a vector space, try to figure out its dimension.)

   (a) The set of $5 \times 5$ matrices $A$ such that $A^T = A$.
   (b) The set of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ that are onto.
   (c) The set of all polynomials with coefficients in the real numbers that have the form $ax^3 + bx$.
   (d) The set of convergent sequences of real numbers.

2. Let $V$ be the subspace of the vector space of continuous functions that is spanned by the set $\{\sin(x), \cos(x), e^x\}$ (you may assume without proof that these three functions are linearly independent). Do the functions $f, g, h$ form a basis for $V$?

   \[ f(x) = 3\sin(x) + 2\cos(x) + e^x \]
   \[ g(x) = \sin(x) + 2\cos(x) \]
   \[ h(x) = 5\sin(x) + 6\cos(x) + e^x \]

3. Let $\mathcal{B} = \{1 + x, x + x^2, 1 + x^2\}, \mathcal{C} = \{1 + x + x^2, 2 + x + x^2, 3x^2\}$ and $T: \mathbb{P}_2 \rightarrow \mathbb{R}^2$ be the linear transformation defined by

   \[ T(p) = \begin{bmatrix} p(2) \\ p(1) + p(3) \end{bmatrix} \]

   (a) Find the matrix for $T$ relative to $\mathcal{B}$ and the standard basis of $\mathbb{R}^2$.
   (b) Find the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$.
   (c) Use your answers to parts (a) and (b) to find the matrix for $T$ relative to $\mathcal{C}$ and the standard basis for $\mathbb{R}^2$.

4. Let $v$ and $u$ be eigenvectors of a matrix $A$ with different eigenvalues. Show that $u + v$ is not an eigenvector of $A$.

5. Let $v_1, \ldots, v_k$ be eigenvectors of a matrix $A$ with distinct eigenvalues. Show that no linear combination of $v_1, \ldots, v_k$ is an eigenvector of $A$.

6. Suppose $A$ is a $5 \times 5$ matrix whose characteristic polynomial is $\lambda^3(\lambda - 1)(\lambda - 2)$. What are the possible values for rank $A$? For which of these values is $A$ diagonalizable?

7. Find a formula for $A^n$ where

   \[ A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \]

8. Mark each of the following statements true or false. For each statement, either give a proof that it is always true or give a counterexample to show it can be false.
(a) Let \( A \) be an \( n \times n \) matrix with only positive eigenvalues such that there is an orthogonal basis for \( \mathbb{R}^n \) consisting of eigenvectors of \( A \). If \( x \) is a nonzero vector then \( x \cdot (Ax) \) is positive.

(b) Every diagonalizable matrix is invertible.

(c) Every invertible matrix is diagonalizable.

(d) Every matrix with a repeated eigenvalue is not diagonalizable.

(e) Let \( V \) be a subspace of \( \mathbb{R}^n \) and \( W \) a subspace of \( V \). Let \( x \) be a vector in \( \mathbb{R}^n \), \( y \) the projection of \( x \) on the subspace \( V \) and \( z \) the projection of \( y \) on the subspace \( W \). Then \( z \) is the projection of \( x \) on \( W \).

(f) If \( v \) and \( u \) are eigenvectors of a matrix \( A \) with distinct eigenvalues then \( v \) and \( u \) are orthogonal.