Math 54 Midterm 2 Review

1. Which of the following are vector spaces? (For an extra challenge, for each set that is a vector space, try to figure out its dimension.)

   (a) The set of 5×5 matrices A such that $A^T = A$.

   (b) The set of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ that are onto.

   (c) The set of all polynomials with coefficients in the real numbers that have the form $ax^3 + bx$.

   (d) The set of convergent sequences of real numbers.

2. Let V be the subspace of the vector space of continuous functions that is spanned by the set $\{\sin(x), \cos(x), e^{e^x}\}$ (you may assume without proof that these three functions are linearly independent). Do the functions f, g, h form a basis for V?

$$f(x) = 3\sin(x) + 2\cos(x) + e^{e^x}$$
$$g(x) = \sin(x) + 2\cos(x)$$
$$h(x) = 5\sin(x) + 6\cos(x) + e^{e^x}$$

3. Let $\mathcal{B} = \{1 + x, x + x^2, 1 + x^2\}, \mathcal{C} = \{1 + x + x^2, 2 + x + x^2, 3x^2\}$ and $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} p(2) \\ p(1) + p(3) \end{bmatrix}.$$

   (a) Find the matrix for T relative to $\mathcal{B}$ and the standard basis of $\mathbb{R}^2$.

   (b) Find the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

   (c) Use your answers to parts (a) and (b) to find the matrix for T relative to $\mathcal{C}$ and the standard basis for $\mathbb{R}^2$.

4. Let $\mathbf{v}$ and $\mathbf{u}$ be eigenvectors of a matrix A with different eigenvalues. Show that $\mathbf{u} + \mathbf{v}$ is not an eigenvector of A.

5. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be eigenvectors of a matrix A with distinct eigenvalues. Show that no linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is an eigenvector of A.

6. Suppose A is a 5×5 matrix whose characteristic polynomial is $\lambda^3(\lambda - 1)(\lambda - 2)$. What are the possible values for rank A? For which of these values is A diagonalizable?

7. Find a formula for A^n where
$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

8. Mark each of the following statements true or false. For each statement, either give a proof that it is always true or give a counterexample to show it can be false.

(a) Let A be an $n \times n$ matrix with only positive eigenvalues such that there is an orthogonal basis for $\mathbb{R}^n$ consisting of eigenvectors of A. If $\mathbf{x}$ is a nonzero vector then $\mathbf{x} \cdot (A\mathbf{x})$ is positive.

(b) Every diagonalizable matrix is invertible.

(c) Every invertible matrix is diagonalizable.

(d) Every matrix with a repeated eigenvalue is not diagonalizable.

(e) Let V be a subspace of $\mathbb{R}^n$ and W a subspace of V. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$, $\mathbf{y}$ the projection of $\mathbf{x}$ on the subspace V and $\mathbf{z}$ the projection of $\mathbf{y}$ on the subspace W. Then $\mathbf{z}$ is the projection of $\mathbf{x}$ on W.

(f) If $\mathbf{v}$ and $\mathbf{u}$ are eigenvectors of a matrix A with distinct eigenvalues then $\mathbf{v}$ and $\mathbf{u}$ are orthogonal.