Midterm 2 Solutions

Math 114S, Winter 2022

Instructions. Turn in your exam on Gradescope by 10 am PST on Saturday, February 26th. Late exams are not accepted, so I advise you to turn it in at least a few minutes early. You may consult the lecture notes, course textbooks or standard websites such as Wikipedia. **You may not attempt to search for specific exam questions online nor may you communicate with anyone besides the instructors of the course about the contents of the exam.** In particular, you should not talk to your fellow students about the exam, except to ask logistics questions, and you may not post exam-related questions on any Q&A websites or forums. If you have any questions about the exam, please make a private post on piazza. There are 50 points in total.

Note: For each problem below, unless otherwise stated, you may use the Axiom of Choice (including any of its consequences which we proved in class).

Short Answer Questions.

Question 1 (4 points)

Compute the rank of the following set: $A = \{\{\langle \mathbb{R}, \mathbb{Z}\rangle, \{\langle \mathbb{Q}, \omega\rangle\}, \langle \mathbb{R}, V_{\omega+2}\rangle\}\}$. You may take as given the calculations of ranks from the Homework 6 solutions. You should show work justifying your answer but you do not need to provide a formal proof that your answer is correct.

Solution: The rank of A is $\boxed{\omega + 9.}$ To see why, first note that it can be shown by transfinite induction that for any $\alpha \in \mathbf{Ord}$, $\text{rank}(V_\alpha) = \alpha$. Using this plus the ranks calculated in homework 6, we can calculate ranks as follows.

Rank	Sets of that rank
ω	ω
$\omega + 1$	$\mathbb{Z}$
$\omega + 2$	$V_{\omega+2}$
$\omega + 3$	
$\omega + 4$	$\mathbb{Q}$
$\omega + 5$	$\mathbb{R}$ $\{\mathbb{Q}\}$ $\{\mathbb{Q}, \omega\}$
$\omega + 6$	$\{\mathbb{R}\}$ $\{\mathbb{R}, \mathbb{Z}\}$ $\{\mathbb{R}, V_{\omega+2}\}$ $\langle \mathbb{Q}, \omega\rangle$
$\omega + 7$	$\langle \mathbb{R}, \mathbb{Z}\rangle$ $\langle \mathbb{R}, V_{\omega+2}\rangle$ $\{\langle \mathbb{Q}, \omega\rangle\}$
$\omega + 8$	$\{\langle \mathbb{R}, \mathbb{Z}\rangle, \{\langle \mathbb{Q}, \omega\rangle\}, \langle \mathbb{R}, V_{\omega+2}\rangle\}$
$\omega + 9$	$\{\{\langle \mathbb{R}, \mathbb{Z}\rangle, \{\langle \mathbb{Q}, \omega\rangle\}, \langle \mathbb{R}, V_{\omega+2}\rangle\}\}$

To see why the table above is correct, note that for each set in the table, its rank according to the table is always the smallest ordinal strictly larger than the ranks of all of its elements.

Common Mistakes: Mistakes made included miscalculating the rank of $V_{\omega+2}$, miscalculating the rank of an ordered pair in terms of the ranks of its elements and overlooking one or more pairs of curly brackets.

Question 2 (8 points)

For any subset $A \subseteq \omega$ and natural number n, say that n is *prime-in-A* if every divisor of n which is contained in A is equal to either 1 or n. For example, if A is the set of numbers greater than 10 then 14 is prime-in-A.

Define a class function $F\colon \mathbf{Ord} \to \mathcal{P}(\omega)$ by transfinite recursion as follows.

Zero case: $F(0) = \omega \setminus \{0, 1\}$

Successor case: $F(\alpha + 1) = \begin{cases} F(\alpha) \setminus \{n\} & \text{if } n \text{ is the least element of } F(\alpha) \text{ which is prime-in-}F(\alpha) \\ F(\alpha) & \text{if no such } n \text{ exists} \end{cases}$

Limit case: $F(\beta) = \bigcap_{\alpha<\beta} F(\alpha).$

For both parts below, you should give a brief justification of your answer, but you do not need to provide a formal proof.

(a) What is $F(\omega + \omega)$?

Solution: $\boxed{F(\omega + \omega) = \varnothing.}$

Let's start by calculating some values of F.

$F(0) = \{2, 3, 4, 5, 6, \ldots\}$	by definition of F
$F(1) = \{3, 4, 5, 6, \ldots\}$	2 is prime (hence prime-in-$F(0)$) and the least element of $F(0)$, so it is removed
$F(2) = \{4, 5, 6, \ldots\}$	likewise, 3 is prime and the least element of $F(1)$
$F(3) = \{5, 6, \ldots\}$	4 is not prime, but it is prime-in-$F(2)$ (because all its proper divisors have already been removed) and it is least in $F(2)$ so it is removed

At this point it seems reasonable to guess that for all $n \in \omega$, $F(n) = \{m \in \omega \mid m \geq n + 2\}$. And indeed, this can be proved by induction on ω. The key point is that if A is a nonempty set of natural numbers which does not contain 0 then the least element of A is always prime-in-A since A does not contain any of its proper divisors (which are all smaller than it).

Thus we have

$$F(\omega) = \bigcap_{n\in\omega} \{m \in \omega \mid m \geq n + 2\} = \varnothing.$$

There are two ways to finish. First, we could note that a simple proof by transfinite induction shows that F is decreasing—i.e. for all $\alpha < \beta$, $F(\beta) \subseteq F(\alpha)$. Second, and even simpler, we can note that since $\omega + \omega$ is a limit ordinal greater than ω, we have

$$F(\omega + \omega) = \bigcap_{\alpha<\omega+\omega} F(\alpha) \subseteq F(\omega) = \varnothing$$

and thus $F(\omega + \omega) = \varnothing$.

(b) What is $F(\omega^3)$? Recall that ω^3 denotes the unique ordinal with order type $\omega \times \omega \times \omega$.

Solution: $\boxed{F(\omega^3) = \varnothing.}$

This follows from the same reasoning as part (a). In particular, ω^3 is a limit ordinal greater than ω so $F(\omega^3) \subseteq F(\omega) = \varnothing$.

Comment: As many students realized, this problem was supposed to work a bit differently. I meant to define a more interesting function that would "run out" after ω^2 steps rather than after ω steps, but I didn't proofread the exam very carefully and ended up with a problem that was a bit simpler than intended. However, I didn't want to change the problem once the exam started so I decided to keep it as is.

Question 3 (12 points)

For each set below, write either "countable," "continuum" or "other" to indicate, respectively, that the set is countable, has cardinality $|2^\omega|$ or that neither of those two possibilities hold. You do not need to provide any justification for your answers.

(a) The set of surjective functions
$\omega \to \{0, 1, 2, \ldots, 10\}$.

Solution: Continuum. Let A denote the set of surjective functions from ω to $\{01, 2, \ldots, 10\}$. The function

$$f \mapsto \left(n \mapsto \begin{cases} n & \text{if } n \leq 10 \\ f(n-11) & \text{if } n > 10. \end{cases}\right)$$

is an injection from 2^ω to A so $|2^\omega| \leq |A|$. On other hand, A is subset of ω^ω so we have

$$|A| \leq |\omega^\omega| \leq |(2^\omega)^\omega| = |2^{(\omega\times\omega)}| = |2^\omega|$$

and thus by the Cantor-Schroeder-Bernstein theorem, $|A| = |2^\omega|$.

(b) The set of surjective functions $\omega \to \omega$.

Solution: Continuum. Let A be the set of surjective functions from $\omega \to \omega$. The function

$$f \mapsto \left(n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ f(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases}\right)$$

is an injection from 2^ω to A so $|2^\omega| \leq |A|$. On other hand, A is subset of ω^ω so we have as in part (a), we have $|A| \leq |2^\omega|$ and thus by the Cantor-Schroeder-Bernstein theorem, $|A| = |2^\omega|$.

(c) The set of surjective functions $\omega \to 2^\omega$.

Solution: Countable. Since 2^ω is not countable, there are no surjective functions from ω to 2^ω. Thus this set is simply the empty set. Since the empty set is finite, it is countable (according to our definition of "countable" from lecture).

Common Mistakes: This was the most frequently missed part of this question. Most people who got it wrong said the set has size $|2^\omega|$, perhaps missing the significance of the word "surjective."

(d) The set of surjective functions $2^\omega \to 2^\omega$.

Solution: Other. Let A be the set of surjective functions from $2^\omega \to 2^\omega$. We will show below that $|2^{(2^\omega)}| \leq |A|$. By Cantor's theorem, $|2^\omega| < |2^{(2^\omega)}|$ and thus $|\omega| < |2^\omega| < |A|$. Therefore A is not countable and also not the same cardinality as 2^ω.

We will now show that $|2^{(2^\omega)}| \leq |A|$. First, for a function $f\colon \omega \to 2$, define $\mathrm{shift}(f)\colon \omega \to 2$ as the function

$$\mathrm{shift}(f)(n) = f(n+1).$$

Also, for any $m \in \omega$, let C_m denote the constantly m function. Now define a function $2^{(2^\omega)} \to A$ by

$$G \mapsto \left(f \mapsto \begin{cases} \mathrm{shift}(f) & \text{if } f(0) = 0 \\ C_{G(f)} & \text{if } f(0) = 1. \end{cases}\right)$$

This function is the desired injection.

(e) The set of isomorphism classes of finite partial orders which are in $V_{\omega+5}$.

Solution: Countable. The basic idea is that every isomorphism class of a finite partial order can be described with a finite amount of information and thus the set of all such isomorphism classes is countable. The part about $V_{\omega+5}$ is mostly a red herring since every isomorphism class of finite partial orders has a representative in $V_{\omega+5}$. However some such restriction is needed as the isomorphism class of a finite partial order is typically not a set and thus the collection of all of them is clearly not a set.

More formally, note that for any $n \in \omega$, any partial order of size n is isomorphic to a partial order *on* n (by a trick similar to the solution of Question 6 below). Moreover, a partial order on n is just a subset of $n \times n$, which is a finite subset of $\omega \times \omega$. Thus every finite partial order is isomorphic to a partial order of the form $\langle n, A\rangle$ where A is a finite subset of $\omega \times \omega$.

Let $\mathrm{Fin}(\omega \times \omega)$ denote the set of finite subsets of $\omega \times \omega$. Then there is a surjective function from $\omega \times \mathrm{Fin}(\omega \times \omega)$ to the set of isomorphism classes of finite partial orders in $V_{\omega+5}$ (it takes each element of $\omega \times \mathrm{Fin}(\omega \times \omega)$ which does represent a partial order to its isomorphism class in $V_{\omega+5}$ and each other element to the isomorphism class of the empty partial order). Finally, it is straightforward to show that $\omega \times \mathrm{Fin}(\omega \times \omega)$ is countable by exhibiting a surjection from $\omega^{<\omega}$ onto it.

(f) $((2^\omega \times 2^\omega) \sqcup 2^\omega)^\omega$.

Solution: Continuum. We can calculate

$$\begin{aligned}
|2^\omega| &\leq |((2^\omega \times 2^\omega) \sqcup 2^\omega)^\omega| && (\text{because } |2| \leq ((2^\omega \times 2^\omega) \sqcup 2^\omega)) \\
&\leq |(2^\omega \sqcup 2^\omega)^\omega| && (\text{because } |2^\omega \times 2^\omega| = |2^\omega|) \\
&= |(2 \times 2^\omega)^\omega| \\
&= |2^\omega \times (2^\omega)^\omega| && (\text{because } |(A \times B)^C| = |A^C \times B^C|) \\
&= |2^\omega \times 2^{(\omega\times\omega)}| \\
&= |2^\omega \times 2^\omega| \\
&= |2^\omega|.
\end{aligned}$$

Thus by the Cantor-Schroeder-Bernstein theorem, $|2^\omega| = |((2^{\omega\omega}) \sqcup 2^\omega)^\omega|$.

Long Answer Questions.

Question 4 (8 points)

If $f, g \colon \omega \to \omega$, then f *dominates* g if for all $n \in \omega$, $g(n) \leq f(n)$. If A is a set of functions from ω to ω then A is *dominating* if for every function $g \colon \omega \to \omega$ there is some $f \in A$ which dominates g. Prove that there is no countable dominating set of functions from ω to ω.

Solution: Suppose A is a countable set of functions $\omega \to \omega$. We will show that A is not a dominating set. Since A is countable, either $A = \varnothing$ or we can enumerate the elements of A in an infinite list $f_0, f_1, f_2, \ldots$ (more formally, there is a surjection $F \colon \omega \to A$ and f_n denotes $F(n)$).

If $A = \varnothing$ then A cannot be a dominating set, for example because no function in A dominates the identity function I_ω.

So let's suppose that there is some sequence of functions $f_0, f_1, f_2, \ldots$ such that $A = \{f_n \mid n \in \omega\}$. We now aim to construct a function not dominated by any element of A. To that end, define a function $g \colon \omega \to \omega$ by

$$g(n) = f_n(n) + 1.$$

We will now show g is not dominated by any element of A and thus A is not a dominating set. Let f be an element of A. Then for some $n \in \omega$, $f = f_n$ and so we have

$$g(n) = f_n(n) + 1 = f(n) + 1 > f(n).$$

Therefore f does not dominate g.

Question 5 (10 points)

Recall that a *graph* is an ordered pair $\langle V, E\rangle$ consisting of a set V, called the *set of vertices*, and a set E of ordered pairs of elements of V, called the *set of edges*. A subset $U \subset V$ is a *clique*[1] if for every $a, b \in U$, $\langle a, b\rangle \in E$. A clique U is *maximal* if there is no clique U' such that U is a proper subset of U'. Show that every graph has a maximal clique.

Solution: The idea is to form a clique by adding vertices one at a time, adding a vertex at each step as long as doing so will keep our set a clique. We can formalize this process using transfinite recursion.

Let α be some ordinal such that there is a bijection from α to V. For each $\beta < \alpha$, let v_β denote the image of β under this bijection. Now define a function $F \colon \alpha + 1 \to \mathcal{P}(V)$ by transfinite recursion as follows

Zero case: $F(0) = \varnothing$

Successor case: $$F(\beta + 1) = \begin{cases} F(\beta) \cup \{v_\beta\} & \text{if } F(\beta) \cup \{v_\beta\} \text{ is a clique.} \\ F(\beta) & \text{otherwise.} \end{cases}$$

Limit case: $$F(\beta) = \bigcup_{\gamma < \beta} F(\gamma).$$

Define $U = F(\alpha)$. We need to show that U is a clique and that it is maximal.

[1]This definition of graph and clique are slightly different from the standard definitions—normally cliques are only defined for undirected graphs without self-loops. Note that in the definition above, if U is a clique and $a, b \in U$ then we must have all of $\langle a, a\rangle, \langle a, b\rangle$ and $\langle b, a\rangle$ in U. I used this definition to keep the definitions as short as possible.

Claim 1. For all $\beta, \gamma \leq \alpha$, if $\beta \leq \gamma$ then $F(\beta) \subseteq F(\gamma)$.

Proof. Fix $\beta \leq \alpha$ and proceed by transfinite induction on γ.

Zero case: If $\gamma = 0$ then $F(\gamma) = \varnothing$ is a subset of every set.

Successor case: Suppose that the conclusion holds for γ and we will show it holds for $\gamma + 1$. There are two cases. Either $\gamma < \beta$ or $\beta \leq \gamma$ and $F(\beta) \subseteq F(\gamma)$.

In the first case, either $\gamma + 1 < \beta$ or $\gamma + 1 = \beta$. In either case we are done.

In the second case, note that by the definition of F, $F(\gamma) \subseteq F(\gamma + 1)$ and so $F(\beta) \subseteq F(\gamma)$ implies $F(\beta) \subseteq F(\gamma + 1)$. □

Claim 2. For all $\beta \leq \alpha$, $F(\beta)$ is a clique.

Proof. We will prove this by transfinite induction on β.

Zero case: $F(0) = \varnothing$ satisfies the definition of clique vacuously.

Successor case: Assume that $F(\beta)$ is a clique and we will show that $F(\beta + 1)$ is a clique. There are two cases: either $F(\beta) \cup \{v_\beta\}$ is a clique or it's not. If it is, then $F(\beta + 1) = F(\beta) \cup \{v_\beta\}$ is a clique by assumption. If it's not then $F(\beta + 1) = F(\beta)$ is a clique by the inductive hypothesis.

Limit Case: Assume that β is a limit ordinal and for all $\gamma < \beta$, $F(\gamma)$ is a clique. We want to show that $F(\beta) = \cup_{\gamma<\beta} F(\gamma)$ is a clique. Let $a, b \in F(\beta)$. Thus for some $\gamma_1, \gamma_2 < \beta$, $a \in F(\gamma_1)$ and $b \in F(\gamma_2)$. Without loss of generality, suppose $\gamma_1 \leq \gamma_2$. Then by Claim 1, $F(\gamma_1) \subseteq F(\gamma_2)$ and thus a and b are both in $F(\gamma_2)$. Since $F(\gamma_2)$ is a clique by the inductive hypothesis, $\langle a, b\rangle \in E$. Thus $F(\beta)$ is a clique. □

Note that Claim 2 implies $U = F(\alpha)$ is a clique. So all that remains is to show it is maximal.

Claim 3. U is maximal.

Proof. Suppose $U' \subseteq V$ is a proper superset of U. We will show that U' is not a clique.

Since U is a proper subset of U', there is $\beta < \alpha$ such that v_β is in U' but not U. Note that $\beta + 1 \leq \alpha$ and so by Claim 1, $F(\beta + 1) \subset F(\alpha) = U$. And since v_β is not in U, this implies that v_β is not in $F(\beta + 1)$. Thus by definition of F, $F(\beta) \cup \{v_\beta\}$ is not a clique. In other words, there are a and b in $F(\beta) \cup \{v_\beta\}$ such that $\langle a, b\rangle \notin E$. Finally, note that since $F(\beta) \subseteq U \subset U'$ and $v_\beta \in U'$, $F(\beta) \cup \{v_\beta\} \subset U'$ and thus $a, b \in U'$ so U' is not a clique. □

Common Mistakes: Most people had the right idea for this question: use transfinite recursion and pick vertices one at a time. However, there were several mistakes both in implementing this idea and in proving it correct. Here are some of the mistakes:

- Several people tried to set $F(0) = \{v\}$ for some vertex v. However, there is no guarantee that the graph has any non-empty cliques so this does not always work. In particular, it could be that for every vertex $v \in V$, $\langle v, v\rangle \notin E$ and thus v cannot be part of any clique.
- Several people wrote definitions that excluded all vertices of the form v_β for β a limit ordinal. Note that in the definition above, on step $\beta + 1$ we try to add v_β rather than $v_{\beta+1}$ and on limit steps we don't add any new vertices at all. This means that every vertex, including

limit vertices, get taken care of at successor steps. If you change the successor steps to try to take add $v_{\beta+1}$ instead of v_β then to make sure limits are still taken care of correctly you need to change the definition of F in the limit case.

- Several people claimed without proof various things about the function F. For example that $F(\beta)$ is a clique for all $\beta \leq \alpha$. Even if you think this is too obvious to prove, it is a good idea to at least mention that it can be proved, for example "this can be proved using transfinite induction on β."
- People who did not properly take care of vertices at limit ordinals typically also had mistakes in their proofs. Usually these mistakes amounted to implicitly assuming that every vertex $v \in V$ was of the form $v_{\beta+1}$ for some β.
- This is not a mistake per se, but some people produced proofs that were quite difficult to read. For example, proofs that launched into the proof of some claim or lemma without clearly stating what was being proved, proofs by transfinite induction that only mentioned the word "induction" halfway through in the middle of the inductive step, proofs by transfinite induction where the inductive hypothesis seemed to change halfway through the proof and so on. People also sometimes introduced new notation without explaining it. Keep in mind that proof writing is still a form of writing and the goal is to clearly communicate your ideas to another person. Thus you should try to write proofs in such a way that anyone reading your proof can easily follow your writing. This includes doing things like: clearly sign-posting what you are proving in each major section of the proof, clearly stating your induction hypothesis when using induction and explaining any new notation or terminology. It also includes structuring your proof so that a reader who is merely skimming it can quickly find the major steps of your proof without having to go through all the details first.

Question 6 (8 points)

Graphs $\langle V, E\rangle$ and $\langle V', E'\rangle$ are *isomorphic* if there is a bijection $f\colon V \to V'$ such that for all $u, v \in V$,

$$\langle u, v\rangle \in E \iff \langle f(u), f(v)\rangle \in E'.$$

Without using the Axiom of Foundation, show that for every graph $\langle V, E\rangle$, there is some $\alpha \in \mathbf{Ord}$ such that V_α contains a graph isomorphic to $\langle V, E\rangle$.

Solution: The idea is just that for any graph $\langle V, E\rangle$, we can pick some ordinal which is in bijection with V and then use the bijection to "copy over" the graph structure to the ordinal.

Let's do this slightly more formally. Let $\langle V, E\rangle$ be a graph. We showed in class that the Axiom of Choice implies that there is some ordinal α such that $|V| = |\alpha|$. Let $f\colon V \to \alpha$ be a bijection witnessing that $|V| = |\alpha|$. Define $E' \subset \alpha \times \alpha$ by

$$E' = \{\langle f(u), f(v)\rangle \mid \langle u, v\rangle \in E\}.$$

Note that $\langle V, E\rangle$ is isomorphic to $\langle \alpha, E'\rangle$ via f and that $\langle \alpha, E'\rangle \in V_{\alpha+5}$ so we are done.

Comment: Here's the broader significance of this question. In set theory it is common to adopt axioms more or less saying that every set is built up only out of other sets—in ZFC this is the purpose of the Axiom of Foundation. There are several reasons set theorists like to do this, including perhaps because it gives a nice and tidy picture of the universe of sets. However, in the "real world" (and perhaps even in the "real world" of pure mathematics), not everything is a set and not every set only contains other sets. Thus to many people, building this assumption into set theory may seem a little crazy.

Here is the standard response from set theorists. First, we can define a class of sets such that each set in the class only contains other sets in the class as elements. This class is often referred to as the class of "hereditary sets" and, in our development of set theory, is equal to $\cup_{\alpha \in \mathbf{Ord}} V_\alpha$. Second, every mathematical structure is structurally identical to one which is only built out of these sorts of sets. Thus adding an axiom saying that *all* sets are hereditary sets does not actually change which mathematical objects can be constructed. So, sure, it may be artificial to consider only "hereditary sets" but it will not cause any practical problems to do so.

This question asks you to show that if we are just interested in graphs and we interpret "structurally identical" to mean "isomorphic to" then the set theorists' claim is true. Note, however, that the proof above requires both the Axiom of Choice and (implicitly) the Axiom of Replacement which are often seen as the most questionable of the axioms of ZFC. In fact, it is not possible to prove the claim without using both of those axioms. On the other hand, most mathematicians seem happy to accept the Axiom of Choice and do not seem bothered by the Axiom of Replacement (assuming they have heard of it) so maybe this is not such a big deal.