Midterm 1

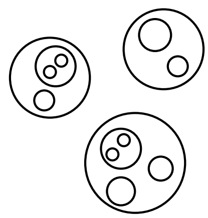
Math 114S, Winter 2022

Instructions. Turn in your exam on Gradescope by 10 am PST on Saturday, January 29th. Late exams are not accepted, so I advise you to turn it in at least a few minutes early. You may consult the lecture notes, course textbooks or standard websites such as Wikipedia. You may not attempt to search for specific exam questions online nor may you communicate with anyone besides the instructors of the course about the contents of the exam (this includes your fellow classmates and also all online Q&A websites, etc). If you have any questions about the exam, please make a private post on piazza. There are 50 points in total.

Short Answer Questions.

Question 1 (5 points)

Consider the drawing below.



Let X be the set of circles in the drawing and define a binary relation $\sim$ on X by

$$x \sim y \iff x \text{ and } y \text{ contain the same number of circles.}$$

The relation $\sim$ is actually an equivalence relation on X (you do not need to prove this). How many equivalence classes does it have and how many elements does each equivalence class have? You do not need to justify your answer.

Question 2 (5 points)

Using the definitions of ω and $\mathbb{Q}$ given in class, is there some natural number n such that

$$\underbrace{\bigcup \cdots \bigcup}_{n \text{ times}} \mathbb{Q} = \omega?$$

If so, what is it? You should briefly justify your answer, but you don't need to formally prove it is correct.

Hint: First try to figure out if there is some number of times you can take the union of $\mathbb{Q}$ to get $\mathbb{Z}$.

Question 3 (5 points)

Rewrite the formula $\langle 1, 1\rangle \in x$ in the language of set theory. You do not need to justify your answer.

In other words, find a formula with one free variable, x, which only contains variables, quantifiers ($\forall$ and $\exists$), logical symbols ($\neg, \wedge, \vee, \implies, \iff$), parentheses and the symbols $\in$ and $=$ and which is equivalent to the formula $\langle 1, 1\rangle \in x$.

Question 4 (12 points)

Mark each of the following as True or False. You do not need to provide any justification for your answers.

(a) There is a set containing all binary relations with domain $\varnothing$.

(b) Russell's paradox does not depend on the Axiom of Extensionality.

(c) Suppose X is a set, $f\colon X \to X$ is a function and A is a subset of X. Define $H\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by $H(B) = B \setminus f[B]$. If $h\colon \omega \to \mathcal{P}(X)$ is the function given by the Recursion Theorem, satisfying

$$h(0) = A \text{ and for all } n \in \omega,\ h(n+1) = H(h(n)),$$

then $h(100) = h(101)$.

(d) According to our definition of $\mathbb{R}$ in class, the set $\{q \in \mathbb{Q} \mid \exists p \in \mathbb{Q}\,(p \times q < 2)\}$ is a real number.

Long Answer Questions.

Question 5 (8 points)

Let X be a set and A and B be subsets of X. Define a function $G\colon \{0,1\}^X \to \{0,1\}^A \times \{0,1\}^B$ by

$$G(f) = (f \restriction A, f \restriction B).$$

Prove that G is surjective if and only if $A \cap B = \varnothing$.

Recall that $\{0,1\}^X$ denotes the set of functions from X to $\{0,1\}$ and $f \restriction A$ denotes the restriction of f to A.

Question 6 (15 points)

In class we claimed that all known math can be formalized within set theory. Suppose that one day you meet some aliens from the planet Ksnadge who are skeptical of this claim. They tell you about a mathematical object that is very important to their development of mathematics—something they call the "Owezrd system." The Owezrd system consists of a set X, an element $a \in X$ and functions $L\colon X \to X$ and $R\colon X \to X$ which satisfy the following axioms:

(1) $a \notin \mathrm{range}(L)$ and $a \notin \mathrm{range}(R)$

(2) R and L are injective

(3) $\mathrm{range}(L)$ and $\mathrm{range}(R)$ are disjoint

(4) For all $A \subseteq X$, if $a \in A$ and A is closed under R and L (i.e. for all $x \in A$, $R(x) \in A$ and $L(x) \in A$) then $A = X$.

You want to show the Ksnadgeans that it is possible to construct an object that satisfies the axioms of the Owezrd system within set theory. In this problem you will see how to do this.

(a) Let X be the set of functions of the form $f\colon n \to \{0,1\}$ for some $n \in \omega$. Show formally that we can form this set in ZFC set theory (recall that with our definition of ω, $0 = \varnothing$ and $n+1 = \{0,1,\ldots,n\}$ for any $n \in \omega$).

(b) Let $a = \varnothing$ be the empty function and let $L\colon X \to X$ and $R\colon X \to X$ be the functions

$$L(f) = f \cup \{\langle \mathrm{dom}(f), 0\rangle\}$$
$$R(f) = f \cup \{\langle \mathrm{dom}(f), 1\rangle\}.$$

Prove that $\mathrm{range}(L)$ and $\mathrm{range}(R)$ are disjoint. You may use without proof any standard facts about the natural numbers (i.e. ω).

Comment. If you are confused about the definition of L and R, note that if $f\colon n \to \{0,1\}$ then $L(f)$ and $R(f)$ are both functions from $n+1 = \{0,1,\ldots,n\}$ to $\{0,1\}$, they both agree with f on all inputs less than n, $L(f)(n) = 0$, and $R(f)(n) = 1$.

(c) Prove that $\langle X, a, L, R\rangle$ as defined in parts (a) and (b) of this question satisfies axiom (4) of the Owezrd system.