Final Exam

Math 114S, Winter 2022

Instructions. Please carefully read the instructions for each section. Also, for each problem on this exam, unless explicitly stated otherwise, you may use all axioms of ZFC. There are 100 points in total.

Short Answer Questions.

For each question in this section, you do not need to give any justification for your answers.

Question 1 (16 points)

Mark each of the following as True or False.

(a) Without making any other changes to our development of mathematics within ZFC, we could have defined an ordered pair as a function with domain $\{\varnothing, \{\varnothing\}\}$.

(b) For every set x, $\text{rank}(\text{tc}(x)) = \text{rank}(x)$, where $\text{tc}(x)$ denotes the transitive closure of x.

(c) There is an ordinal $\alpha \geq \omega$ such that $\alpha^\alpha = \alpha$ (where α^α here is referring to ordinal exponentiation, not cardinal exponentiation).

(d) If M, A, and B are sets such that $A, B \in M$ and $M \models$ "$|A| = |B|$" then $|A| = |B|$.

Question 2 (5 points)

Write the formula "R is a binary relation" in the language of set theory—i.e. using only variables, logical symbols and the symbols $\in$ and $=$.

Question 3 (5 points)

Let $0_\mathbb{R}$ denote the additive identity of the real numbers. Using the definition of $\mathbb{R}$ that we gave in class, list all elements in the transitive closure of $0_\mathbb{R}$.

Examples and Constructions.

For each question in this section, provide the example or construction requested. You do not need to provide any justification for your answers.

Question 4 (8 points)

One day you meet some aliens from the planet Orbifulx, and you learn that everything in their development of mathematics revolves around circles. Therefore, rather than either ordered or unordered pairs, they prefer to consider triplets of objects arranged in a circle, which we will refer to as "circular triples." Two circular triples are considered the same if one can be rotated (but not reflected) so that it is equal to the other. Your task is to convince the Orbifulxians that their concept of "circular triples" can be constructed within set theory.

To be more precise, let $\circ(x, y, z)\circ$ denote the circular triple consisting of x, y, and z arranged in a circle so that if we go around the circle in clockwise order, starting from x, we will encounter x, y and z in that order. So $\circ(x, y, z)\circ = \circ(y, z, x)\circ$ no matter what x, y and z are, but $\circ(x, y, z)\circ = \circ(y, x, z)\circ$ if and only if at least two of x, y, z are equal. Explain how to define $\circ(x, y, z)\circ$ for all sets x, y, z such that it will behave as the Orbifulxians expect.

Question 5 (8 points)

Suppose F is a class function defined on the ordinals and α is an ordinal. Say that F *stabilizes at* α if α is the least ordinal such that for all $\beta > \alpha$, $F(\beta) = F(\alpha)$. Give an example of a class function $F\colon \mathbf{Ord} \to \{0, 1\}$ which does not stabilize at any ordinal.

Question 6 (8 points)

Recall the definition of *stabilizes at* from the previous question. Given any function $g\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, define a class function $F_g\colon \mathbf{Ord} \to \mathcal{P}(\omega)$ by transfinite recursion as follows:

$$\begin{aligned} \text{Zero case:} \quad & F_g(0) = \omega \\ \text{Successor case:} \quad & F_g(\alpha+1) = g(F_g(\alpha)) \\ \text{Limit case:} \quad & F_g(\beta) = \bigcap_{\alpha<\beta} F_g(\alpha). \end{aligned}$$

Give an example of a function $g\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that F_g stabilizes at ω^2 and for all $A \subseteq \omega$, $g(A) \subseteq A$.

Long Answer Questions.

For each question in this section, provide a complete proof.

Question 7 (12 points)

Let A denote the set of all functions $\omega \to \omega$. Let $\sim$ be the binary relation on A defined by

$$f \sim g \iff \{n \in \omega \mid f(n) \neq g(n)\} \text{ is finite.}$$

Note that $\sim$ is an equivalence relation on A (you do not need to prove this). Show that $A/{\sim}$ is uncountable.

Question 8 (10 points)

Without using the Axiom of Choice, show that for any infinite sets A and B, $|A \times B| \leq |A^B|$.

Question 9 (12 points)

Show that the Axiom schema of Separation is not provable from the Axioms of Extensionality, Empty Set and Powerset. In other words, show that there is some instance of the Axiom of Separation that is not provable from those three Axioms.

Question 10 (16 points)

Recall that a *graph* is simply a pair consisting of a set V, called the *set of vertices*, and a set $E \subseteq V \times V$, called the *set of edges*. If x is a set, the *membership graph* of x is the graph whose set of vertices is x and whose set of edges is the set $\{\langle y, z\rangle \mid y, z \in x \text{ and } y \in z\}$. A graph is *realizable* if it is isomorphic to the membership graph of some transitive set.

A graph is *rankable* if there is some way of assigning ordinals to the vertices such that ordinal assignments always increase along edges—i.e. $G = \langle V, E\rangle$ is rankable if there is some function $f\colon V \to \mathbf{Ord}$ such that for all edges $\langle u, v\rangle \in E$, $f(u) < f(v)$. A graph $G = \langle V, E\rangle$ is *rigid* if for all u, v in V,

$$\{w \in V \mid \langle w, u\rangle \in E\} = \{w \in V \mid \langle w, v\rangle \in E\} \implies u = v.$$

Prove that a graph is realizable if and only if it is rankable and rigid.

Extra Credit Questions.

The following question is optional. If you find a correct solution you will receive two points of extra credit.

Question 11 (2 points (bonus))

Suppose $H\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a continuous function and $f\colon \mathbb{R} \to \mathbb{R}$ is a function (not necessarily continuous) such that for all $x \in \mathbb{R}$,

$$|f(x) - x| \leq H(x) - H(f(x)).$$

Intuitively, you can think of H as the "potential energy" of a point and the equation above says that f can only move x very far if it decreases the potential energy of x a lot. Show that there is some $x \in \mathbb{R}$ such that $f(x) = x$.

Hint: There is a way to solve this question that uses ideas we learned in class.