Combinatorics Worksheet 6: Review and 12-Fold Way

1. Suppose that you have \( n \) employees and need to choose some of them to receive a promotion. In each of the following scenarios, how many ways are there to choose which employees receive a promotion?
   (a) Suppose you can choose any number of employees to receive a promotion.
   (b) Exactly 5 employees must receive a promotion.
   (c) Any number of employees can receive a promotion, but at least one of the employees Alan, Kim, and Cassandra must receive a promotion.
   (d) Any number of employees can receive a promotion, but at most one of the employees Alan, Kim, and Cassandra must receive a promotion.

2. What is the coefficient of \( x^6y^7 \) in \( (3x^2 - y)^{10} \)?

3. Give a combinatorial proof of the fact that for all \( n \) and all \( k \leq n \),
   \[
   \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
   \]

4. How many ways are there to pay your employees if you have $1000 and 5 employees? Assume that you are allowed to pay employees nothing and that you don’t have to spend all $1000. Also assume that you must pay employees in dollar amounts (e.g. you cannot pay someone $4.53).

5. How many ways are there to arrange 20 books on a bookcase with 3 shelves? Assume, as in real life, that books are distinguishable and that the order of the books on each shelf matters.

6. **Challenge Problem:** Find as many interesting patterns as possible in Pascal’s triangle.